Method of boundary value problems for control improvement in nonlinear systems *

A.S. Buldaev[1], *O.V. Morzhin*[1], *D. Khaltar*[2]

[1]Buryat State University, 24a Smolina Str., Ulan-Ude, 670000, Russia

[2]National University of Mongolia

Abstract

The article is devoted to study the problem of optimization of region sustainable development strategy for social-ecological-economical model with the block of innovation; the problem is oriented on practical modeling of concrete regions. The efficient initial approach in the multistage procedure of optimization and in the computing experiments is the turnpike solution which does not depend directly from border conditions.

Key words: social-ecological-economical model, well-being function, turnpike solution, turnpike, optimal solution, derivative problem, rule of an extreme aiming, method of consecutive approach.

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Introduction

We consider optimal control problem with free terminal state:

$$\Phi(u) = \varphi(x(t_1)) + \int_T F(x(t), u(t), t)dt \to \inf, \quad T = [t_0, t_1], \tag{1}$$

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0, \quad u(t) \in U, \quad t \in T, \tag{2}$$

where $x(t) = (x_1(t), ..., x_n(t))$ is a state vector, $u(t) = (u_1(t), ..., u_m(t))$ is a control vector.

The set of admissible controls V consists of piecewise continuous functions, which values belong to compact set $U \subset R^m$. Initial state x^0 and control segment T are given.

Analogously to [1 – 3] the problem (1), (2) is considered under the following conditions:

1) function $\varphi(x)$ is continuously differentiable on R^n, functions $F(x, u, t)$, $f(x, u, t)$ and derivatives $F_x(x, u, t)$, $F_u(x, u, t)$, $f_x(x, u, t)$, $f_u(x, u, t)$ are continuous with respect to assemblage of variables (x, u, t) on the set $R^n \times U \times T$;

2) function $f(x, u, t)$ satisfies the Lipschitz condition with respect to x in $R^n \times U \times T$, where the Lipschitz constant is $L > 0$: $\|f(x, u, t) - f(y, u, t)\| \leq L\|x - y\|$;

3) the family of trajectories of the system (2) is restricted.

These conditions guarantee existing and uniqueness of solution $x(t, v)$, $t \in T$ for the system (2) and for any admissible control $v(t)$, $t \in T$.

Let us form the Pontryagin function with adjoint variable $\psi \in R^n$:

$$H(\psi, x, u, t) = \langle \psi, f(x, u, t) \rangle - F(x, u, t).$$

For admissible control $v \in V$ we designate by $\psi(t, v)$, $t \in T$ a solution of the standard adjoint system

$$\dot{\psi}(t) = -H_x(\psi(t), x(t), u(t), t), \quad t \in T, \quad \psi(t_1) = -\varphi_x(x(t_1)),$$

where $u(t) = v(t)$, $x(t) = x(t, v)$.

Using mapping

$$u^*(\psi, x, t) = \arg\max_{u \in U} H(\psi, x, u, t), \quad \psi \in R^n, \quad x \in R^n, \quad t \in T,$$

the maximum principle for control $u \in V$ is represented in the form

$$u(t) = u^*(\psi(t, u), x(t, u), t), \quad t \in T. \tag{3}$$

The boundary value problem of maximum principle has the following form

$$\dot{x}(t) = f(x(t), u^*(\psi(t), x(t), t), t), \quad x(t_0) = x^0,$$

$$\dot{\psi}(t) = -H_x(\psi(t), x(t), u^*(\psi(t), x(t), t), t), \quad \psi(t_1) = -\varphi_x(x(t_1)).$$

In common case right-hand sides of the boundary value problem are discontinuous with respect to phase variables x, ψ.

1 The improvement boundary value problem

The partial increment of arbitrary vector function $g(y_1, ..., y_l)$ with respect to variables $y_{s_1},\ y_{s_2}, ...$ can be denoted in the form:

$$\Delta_{y_{s_1}+\Delta y_{s_1},\ y_{s_2}+\Delta y_{s_2},\ ...} g(y_1, ..., y_l) =$$
$$= g(y_1, ..., y_{s_1} + \Delta y_{s_1}, ..., y_{s_2} + \Delta y_{s_2}, ..., y_l) - g(y_1, ..., y_l).$$

The increment of the cost functional (1) for admissible controls u^0, v has the following form according to the accepted notation:

$$\Delta_v \Phi(u^0) = \Delta_{x(t_1,v)} \varphi(x(t_1, u^0)) + \int_T \Delta_{x(t,v),\ v(t)} F(x(t, u^0), u^0(t), t) dt. \tag{4}$$

Also we denote $\Delta x(t) = x(t, v) - x(t, u^0)$.

It's considered differentiable vector function $p(t) = (p_1(t), ..., p_n(t))$ with condition

$$p(t_1) = -\varphi_x(x(t_1, u^0)) - q, \tag{5}$$

where value q satisfies the following algebraic equation:

$$\left\langle \varphi_x(x(t_1,u^0)), \Delta x(t_1) \right\rangle + \langle q, \Delta x(t_1)\rangle = \Delta_{x(t_1,v)}\varphi(x(t_1,u^0)). \tag{6}$$

Then the increase of terminal part for the functional in (4) can be written as

$$\Delta_{x(t_1,v)}\varphi(x(t_1,u^0)) = -\langle p(t_1), \Delta x(t_1)\rangle =$$
$$= -\int_T \frac{d}{dt}\langle p(t), \Delta x(t)\rangle dt =$$
$$= -\int_T \left\{ \langle \dot{p}(t), \Delta x(t)\rangle + \langle p(t), \Delta_{x(t,v),\ v(t)} f(x(t,u^0), u^0(t), t)\rangle \right\} dt.$$

Taking into account the obtained correlation we rewrite the increase of functional (4) as

$$\begin{aligned} \Delta_v \Phi(u^0) &= -\int_T \left\{ \langle \dot{p}(t), \Delta x(t)\rangle + \Delta_{x(t,v),\ v(t)} H(p(t), x(t,u^0), u^0(t), t) \right\} dt = \\ &= -\int_T \left\{ \langle \dot{p}(t), \Delta x(t)\rangle + \Delta_{v(t)} H(p(t), x(t,v), u^0(t), t) + \right. \\ &\quad \left. + \Delta_{x(t,v)} H(p(t), x(t,u^0), u^0(t), t) \right\} dt. \end{aligned} \tag{7}$$

It's introduced the modified adjoint system for function $p(t)$, which satisfies (5), (6):

$$\dot{p}(t) = -H_x(p(t), x(t,u^0), u^0(t), t) - r(t), \tag{8}$$

where the variable quantity $r(t) = (r_1(t), ..., r_n(t))$ is defined for every time moment $t \in T$ from the following algebraic equation

$$\begin{aligned} &\left\langle H_x(p(t), x(t,u^0), u^0(t), t), \Delta x(t) \right\rangle + \langle r(t), \Delta x(t)\rangle = \\ &= \Delta_{x(t,v)} H(p(t), x(t,u^0), u^0(t), t). \end{aligned} \tag{9}$$

Then owing to the differential algebraic system (8), (9) for function $p(t)$ with the initial conditions (5), (6) the increasing formula for (4) is the form:

$$\Delta_v \Phi(u^0) = -\int_T \Delta_{v(t)} H(p(t), x(t,v), u^0(t), t) dt. \tag{10}$$

It's considered the improvement problem for control $u^0 \in V$ according to the functional (1): it's necessary to compute control $v \in V$ such as $\Phi(v) \leq \Phi(u^0)$.

It's considered the following differential algebraic system:

$$\dot{x}(t) = f(x(t), u^*(p(t), x(t), t), t), \quad x(t_0) = x^0, \tag{11}$$

$$\dot{p}(t) = -H_x(p(t), x(t, u^0), u^0(t), t) - r(t), \tag{12}$$

$$\begin{gathered}\left\langle H_x(p(t), x(t, u^0), u^0(t), t), x(t) - x(t, u^0)\right\rangle + \\ +\langle r(t), x(t) - x(t, u^0)\rangle = \\ = \Delta_{x(t)} H(p(t), x(t, u^0), u^0(t), t),\end{gathered} \tag{13}$$

$$p(t_1) = -\varphi_x(x(t_1, u^0)) - q, \tag{14}$$

$$\begin{gathered}\langle \varphi_x(x(t_1, u^0)), x(t_1) - x(t_1, u^0)\rangle + \langle q, x(t_1) - x(t_1, u^0)\rangle = \\ = \Delta_{x(t_1)} \varphi(x(t_1, u^0)).\end{gathered} \tag{15}$$

The equation (13) can be solved evidently with regard to $r(t)$ using $x(t)$, $p(t)$. Actually, in case of functions f, F linear with respect to x we have $\langle r(t),\ x(t) - x(t, u^0)\rangle = 0$, $t \in T$. For this case we consider $r(t) = 0$, $t \in T$.

In nonlinear case the variable quantity $r(t)$ can be constructed over next rule. If for some number k the condition $x_k(t) \neq x_k(t, u^0)$ is right, then we can write $r_i(t) = 0$, $i \neq k$, $r_k(t) = \frac{\Delta_x H - \langle H_x,\ \Delta x\rangle}{\Delta x_k}$. If for all numbers i the equality $x_i(t) = x_i(t, u^0)$ is true, then we can write $r(t) = 0$.

Analogously to the indicated rule the equation (15) can be solve evidently with respect to q. Under linear function φ we have $q = 0$. In nonlinear case we have $q = 0$ under $\forall i \;\; x_i(t_1) = x_i(t_1, u^0)$ and write $q_i = 0$, $i \neq k$, $q_k = \frac{\Delta_x \varphi - \langle \varphi_x, \Delta x\rangle}{\Delta x_k}$ under $\exists k \;\; x_k(t_1) \neq x_k(t_1, u^0)$.

Thus, from the differential algebraic system (11) – (15) we can pass to the auxiliary two-point boundary value problem for ordinary differential equations:

$$\dot{x}(t) = f(x(t), u^*(p(t), x(t), t), t), \quad x(t_0) = x^0,$$

$$\dot{p}(t) = -H_x(p(t), x(t, u^0), u^0(t), t) - R(p(t), x(t), t),$$

$$p(t_1) = -\varphi_x(x(t_1, u^0)) - Q(x(t_1)).$$

At that functions $R(p, x, t)$, $Q(x)$ are ambiguously determined in common case.

We assume that boundary value problem (11) – (15) has solution $(x(t),\ p(t))$, $t \in T$ (probably, not unique) and control formed as

$$v(t) = u^*(p(t), x(t), t), \quad t \in T, \tag{16}$$

is piecewise continuous function. Then we have $x(t) = x(t, v)$ and owing to definition of mapping u^* we obtain $\Delta_{v(t)} H(p(t), x(t, v), u^0(t), t) \geq 0$. From this and formula (1) we have $\Delta_v \Phi(u^0) \leq 0$.

The approach considered for improvement of controls can be formalized in the following way. It's introduced the differential algebraic adjoint system

$$\dot{p}(t) = -H_x(p(t), x(t), w(t), t) - r(t), \tag{17}$$

$$\begin{gathered} \langle H_x(p(t), x(t), w(t), t), y(t) - x(t)\rangle + \langle r(t), y(t) - x(t)\rangle = \\ = \Delta_{y(t)} H(p(t), x(t), w(t), t) \end{gathered} \tag{18}$$

with the following boundary conditions

$$p(t_1) = -\varphi_x(x(t_1)) - q, \tag{19}$$

$$\begin{gathered} \langle \varphi_x(x(t_1)), y(t_1) - x(t_1)\rangle + \langle q, y(t_1) - x(t_1)\rangle = \\ = \Delta_{y(t_1)} \varphi(x(t_1)). \end{gathered} \tag{20}$$

By definition we suppose that $r(t) \equiv 0$ if $f,\ F$ are linear with respect to x. Also we define $q = 0$ if φ is linear. In nonlinear cases if we have $y(t) = x(t)$, $t \in T$ then we assume $r(t) = 0$. At that if we have $t = t_1$ then we write $q = 0$.

For other cases we can obtain $r(t)$, q evidently from the algebraic equations analogously. Thus, the system (17) – (20) can be reduced to the auxiliary differential adjoint system (probably, not unique).

We denote as $p(t, u, v)$, $t \in T$ the solution of system (17) – (20) under $x(t) = x(t, u)$, $w(t) = u(t)$, $y(t) = x(t, v)$ for admissible controls u, v.

By definition we have $p(t, u, v) = \psi(t, u)$, $t \in T$.

The formula for the functional (10) increase according to new notation has the following form:

$$\Delta_v \Phi(u^0) = - \int\limits_T \Delta_{v(t)} H(p(t, u^0, v), x(t, v), u^0(t), t) dt.$$

The exit control formed using the rule (16) we can write as

$$v(t) = u^*(p(t, u^0, v), x(t, v), t), \quad t \in T. \tag{21}$$

We denote the set of admissible exit controls for the differential algebraic boundary value problem (11) – (15) in the following way:

$$V(u^0) = \{v \in V : \quad v(t) = u^*(p(t, u^0, v), x(t, v), t), \quad t \in T\}.$$

If we have $u^0 \in V(u^0)$, then we write

$$u^0(t) = u^*(p(t, u^0, u^0), x(t, u^0), t) = u^*(\psi(t, u^0), x(t, u^0), t), \quad t \in T.$$

Id est, control u^0 satisfies the condition (3) of the maximum principle.

Backwards, if u^0 satisfies the maximum principle, then it contents the condition (21) under $v = u^0$. Therefore, we write $u^0 \in V(u^0)$.

Thus, for control u^0 which satisfies the maximum principle the boundary value problem (11) – (15) allows the following solution $x(t) = x(t, u^0)$, $p(t) = \psi(t, u^0)$ always.

So, if the boundary value problem (11) – (15) hasn't a solution, then $u^0(t)$ doesn't satisfy the maximum principle.

Using a solution of the boundary value problem (11) – (15) we can formulate the maximum principle for the problem (1), (2) in the following way.

The maximum principle. For optimality of control $u^0 \in V$ it's necessary that the couple $\big(x(t,u^0), \psi(t,u^0)\big)$ is solution of the boundary value problem (11) – (15).

If the problem (1), (2) is linear with respect to x (functions $f(x,u,t)$, $F(x,u,t)$, $\varphi(x)$ are linear relatively x) then the boundary value problem (11) – (15) can be reduced to two Cauchy problems for adjoint and phase systems. At that, the procedure is equivalent to the x-method well known for nonlocal improvement [2].

2 The modified boundary value problem

With the object of increase for quality of the improvement method we can apply the quadratic phase modification of the cost functional (1) similarly to [2 – 3]. This modification allows us to obtain new optimality conditions strengthening the maximum principle in the considered class of optimal control problems. The modified method can strictly improve any control, which doesn't satisfy the maximum principle.

Let's $(u^0(t),\ x(t,u^0))$, $(u(t),\ x(t,u))$, $t \in T$ are admissible processes in the problem (1), (2). We introduce the modified cost functional

$$\Phi_\alpha(u,u^0) = \Phi(u) + \alpha J(u,u^0), \quad \alpha \geq 0, \tag{22}$$

where $J(u,u^0)$ is average weighted state deviation

$$J(u,u^0) = \frac{1}{2}\int\limits_T \langle B(x(t,u) - x(t,u^0)), \quad x(t,u) - x(t,u^0)\rangle dt,$$

B is nonzero, symmetrical and positively defined matrix ($B \neq 0$, $B^T = B$, $\ B > 0$).

Under the admitted conditions we have $J(v,u^0) > 0$, $v \in V$, $v \neq u^0$.

Let us formulate the problem for improvement of control u^0 with regard to the cost functional Φ_α: it's required to find control $v^\alpha \in V$ such as $\Phi_\alpha(v^\alpha,u^0) \leq \Phi_\alpha(u^0,u^0) = \Phi(u^0)$. Then control $v^\alpha \in V$ provides the decrease for the initial cost functional and it's right the following estimation

$$\Phi(v^\alpha) - \Phi(u^0) \leq -\alpha J(v^\alpha,u^0). \tag{23}$$

The phase modification doesn't change the structure of the problem. So, for construction of the modified improvement method we can use the developed approach.

For the modified cost functional (22) the Pontryagin function with the adjoint variable $\psi \in R^n$ may be represented in the following form:

$$H_\alpha(\psi, x, u, t) = H(\psi, x, u, t) - \frac{1}{2}\alpha\langle B(x - x(t, u^0)), x - x(t, u^0)\rangle.$$

The modified differential algebraic boundary value problem is

$$\dot{x}(t) = f(x(t), u^*(p(t), x(t), t), t), \quad x(t_0) = x^0, \tag{24}$$

$$\dot{p}(t) = -H_x(p(t), x(t, u^0), u^0(t), t) - r(t), \tag{25}$$

$$\begin{gathered} \langle H_x(p(t), x(t, u^0), u^0(t), t), \\ x(t) - x(t, u^0)\rangle + \langle r(t), x(t) - x(t, u^0)\rangle = \\ = \Delta_{x(t)} H(p(t), x(t, u^0), u^0(t), t) - \\ -\tfrac{1}{2}\alpha\langle B(x(t) - x(t, u^0)), \quad x(t) - x(t, u^0)\rangle, \end{gathered} \tag{26}$$

$$p(t_1) = -\varphi_x(x(t_1, u^0)) - q, \tag{27}$$

$$\begin{gathered} \langle \varphi_x(x(t_1, u^0)), x(t_1) - x(t_1, u^0)\rangle + \\ \langle q, x(t_1) - x(t_1, u^0)\rangle = \Delta_{x(t_1)}\varphi(x(t_1, u^0)). \end{gathered} \tag{28}$$

The modified differential algebraic adjoint boundary value problem is

$$\dot{p}(t) = -H_x(p(t), x(t), w(t), t) + \alpha B(x(t) - x(t, u^0)) - r(t), \tag{29}$$

$$\begin{gathered} \langle H_x(p(t), x(t), w(t), t) - \alpha B(x(t) - x(t, u^0)), \quad y(t) - x(t)\rangle + \\ \langle r(t), \quad y(t) - x(t)\rangle = \Delta_{y(t)} H_\alpha(p(t), x(t), w(t), t), \end{gathered} \tag{30}$$

where

$$p(t_1) = -\varphi_x(x(t_1)) - q, \tag{31}$$

$$\langle \varphi_x(x(t_1)), \quad y(t_1) - x(t_1)\rangle + \langle q, y(t_1) - x(t_1)\rangle = \Delta_{y(t_1)}\varphi(x(t_1)). \tag{32}$$

Under admissible controls u, v and $x(t) = x(t, u)$, $w(t) = u(t)$, $y(t) = x(t, v)$ we denote solution of the system (29) – (32) as $p^\alpha(t, u, v)$, $t \in T$.

Of course, we have $p^{\alpha}(t,u^0,u^0)=\psi(t,u^0)$, $t\in T$.

Formula for the cost functional (22) increasing under controls u^0, $v\in V$ is

$$\Delta\Phi_{\alpha}(v,u^0)=-\int_T \Delta_{v(t)}H(p^{\alpha}(t,u^0,v),x(t,v),u^0(t),t)dt,$$

where $\Delta\Phi_{\alpha}(v,u^0)=\Phi_{\alpha}(v,u^0)-\Phi_{\alpha}(u^0,u^0)$.

We assume that solution $(x^{\alpha}(t),p^{\alpha}(t))$, $t\in T$ of the boundary value problem (24) – (28) (not unique, possibly) exists at the segment T and the exit control $v^{\alpha}(t)=u^*(p^{\alpha}(t),x^{\alpha}(t),t)$, $t\in T$ is a piecewise continuous function.

We have $x^{\alpha}(t)=x(t,v^{\alpha})$, $p^{\alpha}(t)=p^{\alpha}(t,u^0,v^{\alpha})$, $t\in T$. At that

$$v^{\alpha}(t)=u^*(p^{\alpha}(t,u^0,v^{\alpha}),x(t,v^{\alpha}),t),\quad t\in T. \tag{33}$$

Then owing to the mapping u^* we obtain

$$\Delta_{v^{\alpha}(t)}H(p^{\alpha}(t,u^0,v^{\alpha}),x(t,v^{\alpha}),u^0(t),t)\geq 0,\quad t\in T.$$

That's why we have $\Delta\Phi_{\alpha}(v^{\alpha},u^0)\leq 0$. Thus, the exit control v^{α}, $\alpha\geq 0$ supplies lack of increase for the cost functional (1) according to the estimation (23).

Under $\alpha=0$ (absence of modification) this method coincides with the first method. Value $\alpha>0$ corresponds to the modification of the first method.

We consider set $V^{\alpha}(u^0)$ as the set of controls at the exit of this modified improvement method. At that the equality (33) is right. Analogously to the non-modified method we can formulate the following statement.

Control $u^0\in V$ satisfies the maximum principle then and only then, when $u^0\in V^{\alpha}(u^0)$ at least one $\alpha\geq 0$.

It's significant that if $u^0\in V$ satisfies the maximum principle (3), then $u^0\in V^{\alpha}(u^0)$ for all $\alpha\geq 0$.

The maximum principle in the context for solution of the boundary value problem (24) – (28) can be formulated in the following way.

The maximum principle. For optimality of control $u^0 \in V$ in the problem (1), (2) it's necessary that the pair $(x(t,u^0), \psi(t,u^0))$ is solution of the boundary value problem (24) – (28) at least one $\alpha \geq 0$.

New intensified necessary optimality condition on basis of the modified improvement method can be formulated analogously to [2 – 3].

Condition A. For optimality of control $u^0 \in V$ in problem (1), (2) it's necessary that the pair $(x(t,u^0), \psi(t,u^0))$ is the unique solution of the boundary value problem (24) – (28) for all $\alpha > 0$.

In fact, if for some $\alpha > 0$ we have $v^\alpha \neq u^0$, $v^\alpha \in V^\alpha(u^0)$, then we obtain strict improvement $\Phi(v^\alpha) < \Phi(u^0)$ on the grounds of the estimation (23).

The maximum principle is the consequence from the condition A obviously. So, the modified method can strictly improve controls, which satisfy the maximum principle and don't content the condition A.

On account of the estimation (23) the modified method under $\alpha > 0$ allows to strictly improve any controls, which don't satisfy the maximum principle.

If the boundary value problems haven't solutions, then control $u^0 \in V$ don't content the maximum principle. Then we need to apply other improvement methods.

3 Examples

Example 1 (*improvement of a singular control*). It's considered the following optimal control problem [4, page 220]:

$$\Phi(u) = x_2(1) \to \inf, \quad T = [0,1],$$
$$\dot{x}_1 = u, \quad \dot{x}_2 = -x_1^2, \quad x_1(0) = 0, \quad x_2(0) = 0, \quad |u(t)| \leq 1, \quad t \in T.$$

There are the Pontryagin function $H = p_1 u - p_2 x_1^2$ and the mapping $u^*(p,x,t) = \mathrm{sign} p_1$.

Control $u^0(t) \equiv 0$ is singular. It doesn't satisfy well-known necessary optimality condition of the second-order [1]. We have $x_1(t,u^0) = x_2(t,u^0) = 0$, $t \in T$, $\Phi(u^0) = 0$. At that $\Delta_x H(p, x(t,u^0), u^0(t), t) = -p_2 x_1^2$, $\Delta x_1(t) = x_1(t)$, $\Delta x_2(t) = x_2(t)$.

The differential algebraic boundary value problem for improvement of control u^0 is

$$\dot{x}_1(t) = \text{sign} p_1(t), \quad \dot{x}_2(t) = -x_1^2(t), \quad \dot{p}_1(t) = -r_1(t), \quad \dot{p}_2(t) = -r_2(t),$$

$$x_1(0) = 0, \quad x_2(0) = 0, \quad p_1(1) = 0, \quad p_2(1) = -1,$$

$$r_1(t)x_1(t) + r_2(t)x_2(t) = -p_2(t)x_1^2(t).$$

If $x_i(t) = x_i(t,u^0) = 0$, $i = 1,2$, then we assume $r_i(t) = 0$, $i = 1,2$.

If $x_1(t) \neq 0$, then we assume $r_2(t) = 0$. We have $r_1(t)x_1(t) = -p_2(t)x_1^2(t)$ or $r_1(t) = -p_2(t)x_1(t)$.

If $x_1(t) = 0$, $x_2(t) \neq 0$, then we assume $r_1(t) = 0$. We obtain $r_2(t)x_2(t) = 0$ or $r_2(t) = 0$.

So, we have obvious functions $R_1(p,x,t) = -p_2 x_1$, $R_2(p,x,t) \equiv 0$ for the auxiliary differential boundary value problem, which reduces to the following simplified problem:

$$\dot{x}_1 = \text{sign} p_1(t), \quad \dot{p}_1(t) = -x_1(t), \quad x_1(0) = 0, \quad p_1(1) = 0.$$

We assume that $p_1(0) > 0$. Then $\dot{x}_1(t) = 1$, $x_1(t) = t$, $p_1(t) = \frac{1}{2}(1 - t^2)$ under $t \in [0,1]$. Hence, the boundary value problem allows solution with the exit control $v(t) \equiv 1$. At that it's right $\Phi(v) = -\frac{1}{3} < \Phi(u^0) = 0$.

We assume that $p_1(0) < 0$. Then $\dot{x}_1(t) = -1$, $x_1(t) = -t$, $p_1(t) = \frac{1}{2}(t^2 - 1)$ under $t \in [0,1]$. Thus, the boundary value problem allows the second solution with the exit control $v(t) \equiv -1$. Also we have $\Phi(v) = -\frac{1}{3} < \Phi(u^0) = 0$.

Further, we assume that $p_1(t) \equiv 0$ under $t > 0$. Then it's right $x_1(t) \equiv 0$. We obtain the trivial solution of the boundary value problem. This solution is the obligatory solution of the boundary value problem for the maximum principle.

As a result, the improvement procedure gives us two improved controls $v(t) \equiv \pm 1$.

Example 2 (*effect of modification*). It's considered the following problem [5, pages 57 – 58]:

$$\Phi(u) = \int_0^{\pi} \big(u^2(t) - x^2(t)\big)\, dt \to \inf,$$

$$\dot{x}(t) = u(t), \quad x(0) = 0, \quad u(t) \in R, \quad t \in T = [0, \pi].$$

In this case we have $H = \psi u - u^2 + x^2$, $\dot{\psi}(t) = -2x(t)$, $\psi(\pi) = 0$. The maximinimizing mapping is $u^* = \frac{1}{2}\psi$.

We try to improve the initial control $u^0(t) = 0$, $t \in T$ with the corresponded trajectories $x(t, u^0) = 0$, $\psi(t, u^0) = 0$, $t \in T$. This control is singular.

The boundary value problem without modification (under $\alpha = 0$) is

$$\dot{x}(t) = \frac{1}{2}p(t), \quad x(0) = 0, \quad \dot{p}(t) = -r(t), \quad p(\pi) = 0, \quad r(t)x(t) = x^2(t).$$

If $x(t) = 0$, then we assume $r(t) = 0$ by definition. If $x(t) \neq 0$, then we obtain $x(t) = r(t)$.

As a result the obvious function is $R(p, x, t) = x$ and the boundary value problem can be reduced to the following problem:

$$\dot{x}(t) = \frac{1}{2}p(t), \quad x(0) = 0, \quad \dot{p}(t) = -x(t), \quad p(\pi) = 0.$$

This boundary value problem has the unique trivial solution and the corresponding control is u^0. So, the non-modified procedure of improvement don't improve control u^0 strictly.

We modified the method introducing the following cost functional (under $B = 2E$):

$$\Phi_\alpha(u) = \int_0^{\pi} \big(u^2(t) - x^2(t)\big)\, dt + \alpha \int_0^{\pi} x^2(t) dt, \quad \alpha > 0.$$

In this case we have $H_\alpha = pu - u^2 + (1 - \alpha)x^2$. The boundary value problem in the modified improvement method is

$$\dot{x}(t) = \frac{1}{2}p(t), \quad x(0) = 0, \quad \dot{p}(t) = -r(t), \quad p(\pi) = 0, \quad r(t)x(t) = (1 - \alpha)x^2(t).$$

The mapping R is $R(p,x,t) = (1-\alpha)x$. Consequently, we obtain the auxiliary boundary value problem

$$\dot{x}(t) = \frac{1}{2}p(t), \quad x(0) = 0, \quad \dot{p}(t) = -(1-\alpha)x(t), \quad p(\pi) = 0.$$

Under $\alpha = \frac{1}{2}$ this boundary value problem has the trivial solution and the non-trivial solution $x(t) = C\sin\frac{t}{2}$, $p(t) = C\cos\frac{t}{2}$, $t \in T$, $C \neq 0$. Thus, the modified procedure gives us the zero exit control and the non-zero exit control: $v(t) = \frac{C}{2}\cos\frac{t}{2}$, $t \in T$. At the same time owing to the estimation (23) we have the strict improvement of control u^0 and we have $\Delta_v\Phi(u^0) = -\frac{1}{2}\int\limits_0^\pi x^2(t)dt = -\frac{C^2}{4}\pi < 0$ for the initial cost functional.

So, the modification of the improvement method gives us strict improvement for the singular control, which can not be improved with help of the non-modified procedure.

Example 3 (*nonlinear terminal cost functional*). Let us consider problem from [4, page 214]:

$$\Phi(u) = \sin\frac{\pi x(1)}{2} \to \inf, \quad T = [0,1],$$

$$\dot{x}(t) = u(t), \quad x(0) = 2, \quad |u(t)| \leq 1, \quad u^0(t) = -1, \quad t \in T.$$

We have $x(t,u^0) = -t+2$ and value $\Phi(u^0) = 1$.

The Pontryagin function is $H(p,x,u,t) = \psi u$. The standard adjoint system

$$\dot{\psi}(t) = 0, \quad \psi(1) = -\frac{\pi}{2}\cos\frac{\pi x(1)}{2}$$

has solution $\psi(t,u^0) \equiv 0$.

Since we have $H(\psi(t,u^0), x(t,u^0), u^0(t), t) \equiv 0$, $H_u(\psi(t,u^0), x(t,u^0), u^0(t), t) \equiv 0$ и $\Delta_u H(\psi(t,u^0), x(t,u^0), u^0(t), t) \equiv 0$, then control $u^0(t) \equiv 0$ is singular at segment T.

The mapping $u^*(p, x, t) = \text{sign} p$. The boundary value problem is

$$\dot{x}(t) = \text{sign} p(t), \quad x(0) = 2,$$

$$\dot{p}(t) = 0, \quad p(1) = -q,$$

$$q(x(1) - 1) = \sin \frac{\pi}{2} x(1) - 1.$$

If $x(1) = 1$, then we assume $q = 0$. If $x(1) \neq 1$ then we assume

$$q = \frac{\sin \frac{\pi}{2} x(1) - 1}{x(1) - 1}.$$

So, function Q is

$$Q(x) = \begin{cases} \frac{\sin \frac{\pi}{2} x - 1}{x - 1}, & x \neq 1; \\ 0, & x = 1. \end{cases}$$

The auxiliary boundary value problem is

$$\dot{x}(t) = \text{sign} p(t), \quad x(0) = 2,$$

$$\dot{p}(t) = 0, \quad p(1) = -Q(x(1)).$$

We have $p(t) = -Q(x(1))$, $t \in T$.

We assume that $p(0) > 0$ (or $Q(x(1)) < 0$). Hence we have the Cauchy problem $\dot{x}(t) = 1$, $x(0) = 2$, which has the following solution: $x(t) = t + 2$, $t \in T$. At that $Q(x(1)) = -1 < 0$. So, the boundary value problem allows solution $x(t) = t + 2$, $t \in T$ with the exit control $v = 1$, which improves the initial control: $\Phi(v) = -1 < \Phi(u^0) = 1$.

Exit control $v = 1$ is singular ($\psi(t, v) \equiv 0$) and it satisfies the second order necessary optimality condition (as was shown in [4, page 214]).

Conclusion

Let us enumerate the main properties of the suggested improvement procedures in the considered class of nonlinear optimal control problems.

1. The laboriousness for improvement is defined on basis of the laboriousness for solution of the special boundary value problem, which is easier then the boundary value problem of the maximum principle in terms of smoothness.

2. For optimal control problem linear with respect to states the improvement procedure is reduced to two Cauchy problems for phase and adjoint systems.

3. Non-local improvement of control, id est initial and improved controls don't connect with any parameter of their closeness.

4. Absence of a procedure of weak or needle-shaped variation of controls, as against to standard local improvement methods.

5. Possibility for improvement of controls, which satisfy the maximum principle, including singular controls. This possibility is due to case of non-uniqueness when we solve the improvement boundary value problem.

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