On the radicals of normed rings in W. I

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Dedicated to the memory of professor D.Ishtseren

Abstract

For any class $\mathcal{M}$ of rings which are normed in a linearly ordered ring W, we give characterizations and prove some properties of normed radicals and we also give characterizations of the Banach algebras $\mathbb{C}^S$ and $(l^\infty(S))$.

In this paper, rings are associative, not necessarily with unity. As usual, $I \triangleleft A$ will denote that I is an ideal in a ring A. We recall that a universal class $\mathcal{M}$ satisfies the following conditions:

(i) $\mathcal{M}$ is closed under homomorphisms;

(ii) $\mathcal{M}$ is hereditary (that is, $I \triangleleft A \in \mathcal{M}$ implies $I \in \mathcal{M}$).

Let us also recall that a (*Kurosh-Amitsur*) *radical* γ in a universal class $\mathcal{M}$ of rings is a class of rings in $\mathcal{M}$ which is closed under homomorphisms, extensions (that is, $I \in \gamma$ and $A/I \in \gamma$ imply $A \in \gamma$), and has the inductive property (that is, if $I_1 \subseteq ... \subseteq I_\lambda \subseteq ...$ is a chain of ideals of a ring $A \in \mathcal{M}$ and each $I_\lambda \in \gamma$, then $\cup I_\lambda \in \gamma$).

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Every ring $A \in \mathcal{M}$ contains a unique largest γ-ideal (that is, an ideal which is in γ), denoted by $\gamma_{\mathcal{M}}(A)$, which is called the $\gamma_{\mathcal{M}}$-*radical* of A. If $\gamma_{\mathcal{M}}$ is a radical, the class

$$S(\gamma_{\mathcal{M}}) = \{A : \gamma_{\mathcal{M}}(A) = 0\}$$

is called the semisimple class of $\gamma_{\mathcal{M}}$. A class $\mathcal{M}_0 \subseteq \mathcal{M}$ of rings is said to be *regular* if every nonzero ideal of a ring in $\mathcal{M}$ has a non-zero homomorphic image in $\mathcal{M}_0$. Starting from a regular (in particular, hereditary) class $\mathcal{M}_0 \subseteq \mathcal{M}$, the *upper radical operator* $U_{\mathcal{M}}$ yields a radical class:

$$\mathcal{U}_{\mathcal{M}}(\mathcal{M}_0) = \{A \in \mathcal{M} : A \text{ has no non-zero homomorphic image in } M_0\}.$$

The fundamental properties of radicals can be found in ([1], [2], [3], [4]).

In what follows, let W be a linearly ordered ring.

Definition 1. *Let A be a ring. A norm on A is a map $\|\cdot\| : A \longrightarrow W$ such that, for each $x, y \in A$,*

(i) $\|x\| \geq 0$; $\|x\| = 0$ *if and only if* $x = 0$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$,

(iii) $\|xy\| \leq \|x\| \; \|y\|$.

A ring which satisfies the conditions (i)-(iii) for some norm in W, is said to be normed in W.

The class of all rings that are normed in W shall be denoted by C_W.

We shall start by presenting some elementary examples.

Examples.

(i) *For any non-empty set S, let $\mathbb{C}^S$ be the set of functions from S into $\mathbb{C}$ where $\mathbb{C}$ is the set of complex numbers. Define pointwise algebraic operations by*

$$(\alpha f + \beta g)(s) = \alpha f(s) + \beta g(s)$$

$$(fg)(s) = f(s)g(s)$$

$$1(s) = 1$$

for each $s \in S$, *each* $f, g \in \mathbb{C}^s$ *and each* $\alpha, \beta \in \mathbb{C}$. Then C^s *is a commutative unital algebra. If we write* $l^\infty(S)$ *for the subset of bounded functions on* S *and define the uniform norm* $|\cdot|_S$ *on* S *by*

$$|f|_S = \sup\{|f(s)| : s \in S\}$$

for any $f \in l^\infty(S)$, *then* $(l^\infty(S), |\cdot|_S)$ *is a unital Banach algebra.*

(ii) *Let* X *be a topological space (for example, think of* $X = \mathbb{R}$*). If we write* $C(X)$ *for the algebra of continuous function on* X, *and* $C^b(X)$ *for the algebra of bounded continuous functions on* X, *then* $(C^b(X), |\cdot|_X)$ *is a unital Banach algebra. Now if we take* Ω *to be a compact space (for example,* $\Omega = [0, 1]$*), then we have* $C^b(\Omega) = C(\Omega)$ *and so* $(C(\Omega), |\cdot|_\Omega)$ *is a unital Banach algebra.*

(iii) *Let* $D = \{z \in \mathbb{C} : |z| < 1\}$ *be the open disc. The disc algebra*

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f \text{ is analytic on } \mathbb{D}\}$$

is a unital Banach algebra.

(iv) *For linear spaces* E *and* F, *the collection* $\Im(E, F)$ *of all linear maps from* E *to* F *is itself a linear space for the standard operations. Now let* E *and* F *be Banach spaces. Then the family* $B(E, F)$ *of all bounded (that is, continuous) linear operators from* E *to* F *is a subspace of* $\Im(E, F)$ *and* $B(E, F)$ *is itself a Banach space for the operator norm given by*

$$\|T\| = \sup\{\|T(x)\| : x \in E,\ \|x\| \leq 1\}.$$

we write $L(E)$ and $B(E)$ for $L(E, E), B(E, E)$, respectively. The product of two operators S and T in $L(E)$ is given by composition:

$$(ST)(x) = (S \circ T)(x) = S(T(x))$$

for any $x \in E$. *Trivially,* $\|ST\| \leq \|S\|\ \|T\|$ *for any* $S, T \in B(E)$ *and* $(B(E), \|\cdot\|)$ *is a unital Banach algebra. The unity of* $B(E)$ *is the identity operator* I_E. *This is* *a non-commutative example. Indeed, if* E *is the finite-dimensional space* $\mathbb{C}^n$ *(say with the Euclidean norm* $\|\cdot\|_2$*), then* $L(E) = B(E)$ *is just the algebra* $M_n(\mathbb{C})$ *of all* $n \times n$ *matrices over* $\mathbb{C}$ *(with the usual identifications).*

(v) *If* W *has a unity, then any ring* A *is normed in* W*. In fact if* $0 \neq a \in A$, *then* $\|a\| = 1$. If $a = 0$, *then* $\|a\| = 0$. *This called the trivial norm.*

Proposition 1. C_W *is a universal class.*

Proof. Clearly, C_W is hereditary.

First we will define a norm on A if $A \cong B$ and B is normed in W. Since A and B are isomorphic, there exists an isomorphism $f : A \to B$ and so we may define, for any $a \in A$, $\|a\| \overset{def}{=} \|f(a)\|$, which shows that A is normed in W.

Let $\overline{A}$ be a homomorphic image of $A \in C_W$. Then we have that $\overline{A} \cong A/I$, where $I = ker\ f = \{a \in A : f(a) = 0\}$. Now $A/I \in C_W$ since, for any element $a \in A$, we can define the norm

$$\|a + I\| = \begin{cases} \|a\| & if\ a \notin I \\ 0 & if\ a \in I \end{cases}$$

Therefore, $\overline{A} \in C_W$.$\square$

We denote by Ass the class of all associative rings and let γ be a radical in Ass. We say that γ is *normed* in W if every ring $A \in \gamma$ is normed in W.

Lemma 2. (Andrunakievitch). If $K \lhd I \lhd A$ and K_A denotes the ideal of A generated by K, then $(K_A)^3 \subseteq K$.

We denote by $\mathcal{L}(\mathcal{M})$ the lower radical generated by $\mathcal{M}$, where $\mathcal{M}$ is a class of rings.

Lemma 3. *Let* A *be a simple ring with unity which is normed in W. The lower radical* $\mathcal{L}(A)$ *in Ass is normed in W.*

Proof. Suppose $B \in \mathcal{L}(A)$. Then B has a nonzero accessible subring A_1 such that

$$B = I_n \triangleright I_{n-1} \triangleright ... \triangleright I_1 \text{ and } I_1 \cong A.$$

Since A has a unity, $A^2 = A$. Therefore I_1 has a unity and $I_1^2 = I_1$. By Lemma 2, $I_1 \triangleleft I_3$ and so $I_1 \triangleleft B$. Thus we can show that $B = A_1 \oplus B_1$, where $A_1 \cong A$. Therefore $B_1 \in \mathcal{L}(A)$. Repeating the procedure, we obtain $B = A_1 \oplus A_2 \oplus ... \oplus A_n \oplus ...$ where $A \cong A_1 \cong A_2 \cong A_n \cong ...$. Hence, for $a \in B$, we can write $a = \sum_{j=1}^{m} c_{i_j}$ $(c_{i_j} \in A_{i_j})$. Now we can define a norm on B by $\|a\| \overset{def}{=} \sum_{j=1}^{n} \|c_{i_j}\|$, because all A_i are normed by Proposition 1. Since B is a direct sum of the A_i, $a = 0$ if and only if $c_{i_j} = 0$. Then, clearly, for any $a \in B$, $\|a\| = 0$ if and only if $a = 0$ and

$$\|a + b\| \leq \|a\| + \|b\| \text{ and } \|ab\| \leq \|a\| \, \|b\|$$

for any $a, b \in B$. Thus every ring $B \in \mathcal{L}(A)$ is normed in W. □

Let V be a universal class of rings containing $\mathbb{Z}^0$ (where $\mathbb{Z}^0$ is the zero ring over the additive group of integers $\mathbb{Z}$), U a universal subclass of V and γ a radical in U.

We denote by $l_V(\gamma)$ the lower radical in V generated by γ (see [5]).

Lemma 4. $\gamma = l_V(\gamma) \cap U$.

Proof. Clearly, $\gamma \subseteq l_V(\gamma) \cap U$. To complete the proof, let $S\gamma$ be the semisimple class of γ in V. If $S\gamma = 0$, then $\gamma = U$ and $\gamma \supseteq l_V(\gamma) \cap U$. If $S\gamma \neq 0$, then $S\gamma$ is a regular class in U and so the upper class $\mathcal{U}_V(S\gamma)$ in U is a radical class. Thus we have

$$l_V(\gamma) \cap U \subseteq l_V(\gamma) \subseteq \mathcal{U}_V(S\gamma).$$

Suppose that $l_V(\gamma) \cap U \nsubseteq \gamma$. Then there exists a nonzero ring $A \in l_V(\gamma) \cap U$ such that $A \notin \gamma$. Hence we have a nonzero homomorphic image $\overline{A}$ of A such that $\overline{A} \in S\gamma$. But, by above, $\overline{A} \in \mathcal{U}_V(S\gamma)$ and so $\overline{A} \in \mathcal{U}_V(S\gamma) \cap S\gamma$; a contradiction. So we must have $l_V(\gamma) \cap U \subseteq \gamma$. □

Lemma 5. *Let U be a universal subclass of a universal class V and let γ be a radical in V. Then $\gamma \cap U$ is a radical in U.*

Proof. Since U and γ are homomorphically closed, $\gamma \cap U$ is homomorphically closed. Let I and A/I be in $\gamma \cap U$ and $A \in U$. Since γ is a radical class, $A \in \gamma$. Thus we have $A \in \gamma \cap U$. Let $I_1 \subseteq ... \subseteq I_\lambda \subseteq ...$ be a chain of ideals of the ring $A \in U$ such that each I_λ is in γ. Then it is easy to see that $A \in \gamma \cap U$. $\Box$

Theorem 6. *Let γ be a radical class in Ass. Then $\gamma \cap C_W = l_{Ass}(\gamma \cap C_W) \cap C_W$.*

Proof. By Lemma 5, $\gamma \cap C_W$ is a radical class in C_W. Hence, using Lemma 4, we get $\gamma \cap C_W = l_{Ass}(\gamma \cap C_W) \cap C_W$. $\Box$

Now, we consider a linearly ordered ring W without nonzero nilpotent elements. We denote by HC_W the class of all rings in C_W such that every nonzero ideal I of $A \in C_W$ has a nonzero element x such that $\|x^n\| = \|x\|^n$, for any natural n.

Theorem 7. *Let $\mathcal{N}$ be Koëthe's nil radical in Ass. Then*

$$\mathcal{U}_{C_W}(HC_W) \supseteq l_{Ass}(\mathcal{U}_{C_W}(HC_W)) \cap C_W = \mathcal{N} \cap C_W.$$

Proof. We need to prove that $\mathcal{U}_{C_W}(HC_W) \supseteq \mathcal{N} \cap C_W$. Let $A \in \mathcal{N} \cap C_W$ and suppose $A \notin \mathcal{U}_{C_W}(HC_W)$. Then A has a nonzero homomorphic image $\overline{A}$ such that $\overline{A} \in HC_W$. By the definition of HC_W, for any nonzero $I \lhd \overline{A}$, I has a nonzero element x such that $\|x^n\| = \|x\|^n$, for all $n \in \mathbb{N}$; thus $\|x\| > 0$ for any $n \in \mathbb{N}$. Since A is a nil ring, $\overline{A}$ is a nil ring and for some natural number m, $x^m = 0$. But $0 = \|x^m\| = \|x\|^m \neq 0$; a contradiction. Thus we have $\mathcal{N} \cap C_W \subseteq U_{C_W}(HC_W)$. The proof is complete, by Theorem 6. $\Box$

Theorem 8. Let W be a linearly ordered ring, without nonzero nilpotent elements and with unity. Then $\mathcal{U}_{C_W}(HC_W) = \mathcal{N}$.

Proof. We claim that $\mathcal{N} \subseteq \mathcal{U}_{C_W}(HC_W)$. By Theorem 7, $\mathcal{N} \cap C_W \subseteq \mathcal{U}_{C_W}(HC_W)$. Since W has a unity, by example (v), all rings in Ass are normed in W, for the trivial norm. Hence $C_W = Ass$ and $\mathcal{N} \subseteq C_W$. Thus $\mathcal{N} = \mathcal{N} \cap C_W \subseteq \mathcal{U}_{C_W}(HC_W)$. Finally, $\mathcal{U}_{C_W}(HC_W) \subseteq \mathcal{N}$. In fact, suppose $\mathcal{U}_{C_W}(HC_W) \nsubseteq \mathcal{N}$. Then there exist a nonzero ring $A \in \mathcal{U}_{C_W}(HC_W)$ such that $A \notin \mathcal{N}$. Hence we have a nonzero homomorphic image $\overline{A}$ of A, such that $\overline{A} \in S(\mathcal{N})$. Then $\overline{A}$ has no nonzero nil ideals. Let $I \triangleleft \overline{A}$. Then there exists $0 \neq x \in I$, which is not nilpotent. We consider the trivial norm $\|\cdot\|$, since W has unity. Then, for any natural n,

$$1 = \|x^n\|,$$

$$\|x\|^n = 1^n = 1$$

and so $\|x\|^n = \|x^n\|$. Hence $\overline{A} \in HC_W$ and $\overline{A} \in \mathcal{U}_{C_W}(HC_W) \cap HC_W = 0$; a contradiction. □

Let $\mathcal{Q}$ denote the Brown-McCoy radical; that is, the upper radical generated by all simple ring with unity.

Theorem 9. *Let U be a universal subclass of Ass. Then $\mathcal{U}_{C_W}(C'_W) = \mathcal{Q} \cap C_W$, where C'_W is the class of all simple rings with unity, in C_W.*

Proof. $\mathcal{U}_{C_W}(C'_W) \subseteq \mathcal{Q} \cap C_W$: Let $A \in \mathcal{U}_{C_W}(C'_W)$. Clearly, $A \in C_W$, and it remains to show that $A \in \mathcal{Q}$. If $A \notin \mathcal{Q}$, then A has a nonzero homomorphic image $\overline{A}$ such that $\overline{A}$ is a simple ring with unity. Since $A \in C_W$, by Proposition 1, $\overline{A} \in C_W$, and also $\overline{A} \in C'_W$. Therefore $\overline{A} \in \mathcal{U}_{C_W}(C'_W) \cap C'_W = 0$; a contradiction. We claim that $\mathcal{Q} \cap C_W \subseteq \mathcal{U}_{C_W}(C'_W)$. Let $A \in \mathcal{Q} \cap C_W$ and suppose $A \notin \mathcal{U}_{C_W}(C'_W)$. Since $A \in C_W$, A has a nonzero homomorphic image $\overline{A}$ in C'_W. Thus $\overline{A} \notin \mathcal{Q}$; a contradiction. □

We consider the rings $\mathbb{C}^S$ in Example (i), and also the Thierrin radical γ_t in Ass, which is the upper radical of all fields. Clearly, every ring $A \in S(\gamma_t)$ is a subdirect sum of fields.

Proposition 10. *If $\mathbb{C}^S$ is normed in a linearly ordered ring W, then $\mathbb{C}^S$ and $l^\infty(S)$ are $\gamma_t \cap C_W$-semisimple and also suddirect sums of fields normed in W. (For example, $\mathbb{C}^S$ and $l^\infty(S)$ are normed rings in $\mathbb{R}$).*

Proof. Let $f \in \mathbb{C}^S$ and put

$$g(x) = \begin{cases} 0, & if \quad f(x) \neq 0 \\ 1 & if \quad f(x) = 0 \end{cases}.$$

Then $g \in \mathbb{C}^S$. Since $\mathbb{C}^S$ is a commutative ring, $f(x)\mathbb{C}^S g(x) = f(x)g(x)\mathbb{C}^S = 0$. Thus $\mathbb{C}^S$ is not a prime ring and hence $\mathbb{C}^S$ is not a simple ring. Now we shall prove that every nonzero ideal I of $\mathbb{C}^S$ has an idempotent. Let $0 \neq f \in I \lhd \mathbb{C}^S$. Suppose that

$$g(x) = \begin{cases} \frac{1}{f(x)}, & if \quad f(x) \neq 0 \\ 0 & if \quad f(x) = 0 \end{cases}.$$

Then

$$e(x) = f(x)g(x) = \begin{cases} 1, & if \quad f(x) \neq 0 \\ 0 & if \quad f(x) = 0 \end{cases}.$$

Since I is an ideal of $\mathbb{C}^S$, $e \in I$ and $e^2 = (fg)^2 = e$ and thus I has an idempotent element. $\mathbb{C}^S$ is a subdirect sum of subdirectly irreducible rings $\mathbb{C}^S/J_i$ with hearts $\overline{I}_i$ having an idempotent, where $\overline{I}_i = I_i/J_i$ for some $I_i \lhd \mathbb{C}^S$. Since $\overline{I}_i$ has an idempotent element, $\overline{I}_i$ is a simple ring. Since simple commutative rings are fields, $\overline{I}_i$ are fields. Thus $\mathbb{C}^S/J_i \equiv \overline{I}_i$. and $\mathbb{C}^S$ is a subdirect sum of fields.

The proof of the case $l^\infty(S)$ is similar. □

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