Projective perturbation method in linear optimal control problems*

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Abstract

A new numerical approach for solving linear optimal control problems is proposed. The approach is based on the perturbation of control improvement condition. The conditions for method convergence are defined.

Keywords: linear optimal control problem, condition of control improvement, perturbation method, convergence.

1 Introduction

In works [1,2] the methods of nonlocal improvement based on the non-standard formulas of the functional increment are constructed in the class of linear and polynomial with respect to state optimal control problems.

Absence of parametrical variations operation at every iteration and possibility of extreme controls improvement cause the high efficiency of the constructed methods.

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spect to t on the set $R^n \times T$. For admissible controls $u(t)$, $t \in T$, the set V of piecewise continuous functions with values in the convex set $U \subset R^m$ is considered. The initial state x^0 and the control interval T are given.

In the problem (1), (2) the Pontryagin's function has the following structure

$$\begin{aligned} H(\psi, x, u, t) &= H_0(\psi, x, t) + \langle H_1(\psi, x, t), u \rangle, \\ H_0(\psi, x, t) &= \langle \psi, b(x, t) \rangle - d(x, t), \quad H_1(\psi, x, t) = A^T(x, t)\psi - a(x, t), \end{aligned}$$

with $\psi \in R^n$ being the adjoint variable.

Let us introduce standard adjoint vector system

$$\dot{\psi}(t) = -H_x(\psi(t), x(t), u(t), t), \quad \psi(t_1) = \varphi_x(x(t_1)),\ t \in T. \tag{3}$$

For admissible control $v \in V$, by $x(t, v)$, $t \in T$, we denote a solution of the system (2) as $u(t) = v(t)$; by $\psi(t, v)$, $t \in T$, we denote a solution of the system (3) as $u(t) = v(t)$, $x(t) = x(t, v)$.

Let P_U be a projection operator on the set U in Euclidean norm:

$$P_U(z) = \arg \min_{w \in U}(\|w - z\|), \quad z \in R^m.$$

For admissible control $u \in V$, let us form the vector-valued function u^α with the parameter $\alpha > 0$ using the relation

$$u^\alpha(\psi, x, t) = P_U\left(u(t) + \alpha H_1(\psi, x, t)\right),\ x \in R^n,\ \psi \in R^n,\ t \in T.$$

In view of fulfillment the Lipschitz condition for the operator P_U, the function u^α is continuous in $(\psi, x) \in R^n \times R^n$ and piecewise continuous with respect to $t \in T$. According to the known projection property, the following inequality exists

$$\langle H_1(\psi, x, t), u^\alpha(\psi, x, t) - u(t) \rangle \geq \frac{1}{\alpha} \|u^\alpha(\psi, x, t) - u(t)\|^2.$$

Using the function u^α the maximum principle for the control $u \in V$ in the problem (1), (2) can be written in the following form

$$u(t) = u^\alpha(\psi(t, u), x(t, u), t), \quad t \in T. \tag{4}$$

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$$u(t) = u^\alpha(\psi(t, u), x(t, u), t), \quad t \in T. \tag{4}$$

Note that, to fulfill the maximum principle, it is sufficient to examine the condition (4), at least for one $\alpha > 0$.

Let us introduce a modified adjoint vector system

$$\dot{p}(t) = -H_x - \frac{1}{2}H_{xx}z, \quad p(t_1) = -\varphi_x - \frac{1}{2}\varphi_{xx}z. \tag{5}$$

For admissible controls u, v designate by $p(t, u, v)$, $t \in T$ – a solution of system (5), for $\psi = p(t)$, $x = x(t, u)$, $u = u(t)$, $z = x(t, v) - x(t, u)$. It is evident that $p(t, u, u) = \psi(t, u)$, $t \in T$.

Using modification (5) of adjoint system in the problem (1), (2), the formula for the increment of the functional without remainder term of the Taylor series expansion can be obtained [2]:

$$\Delta_v\Phi(u) = -\int_T \Delta_{v(t)}H(p(t, u, v), x(t, v), u(t), t)dt,$$

This formula is the basis for construction of nonlocal improvement condition.

3 Control Improvement Condition

Let us set the control improvement problem for $u^0 \in V$ with respect to functional (1): to find a control $v \in V$ satisfying the condition $\Phi(v) \leq \Phi(u^0)$.

Let us consider the improvement boundary-value problem [3] based on the map u^α in problem (1), (2)

$$\dot{x}(t) = f(x(t), u^\alpha(p(t), x(t), t), t), \quad x(t_0) = x^0, \tag{6}$$

$$\begin{aligned}\dot{p}(t) = &- H_x(p(t), x(t, u^0), u^0(t), t)\\ &- \frac{1}{2}H_{xx}(p(t), x(t, u^0), u^0(t), t)(x(t) - x(t, u^0)),\\ p(t_1) = -\varphi_x(x(t_1, u^0)) &- \frac{1}{2}\varphi_{xx}(x(t_1, u^0))(x(t_1) - x(t_1, u^0)).\end{aligned} \tag{7}$$

Let $x^\alpha(t)$, $p^\alpha(t)$, $t \in T$ be a solution of this problem. It is evident that $x^\alpha(t) = x(t, v^\alpha)$, $p^\alpha(t) = p(t, u^0, v^\alpha)$, $t \in T$. Then the output control $v^\alpha(t) = u^\alpha(p^\alpha(t), x^\alpha(t), t)$, $t \in T$ provides lack of increase of target functional with the estimate

$$\Phi(v^\alpha) - \Phi(u^0) \leq -\frac{1}{\alpha} \int_T \left\| v^\alpha(t) - u^0(t) \right\|^2 dt \tag{8}$$

in view of fulfillment the known projection property.

Such a improvement boundary-value problem is considerably easier than the boundary-value problem of maximum principle, and is reduced to two Cauchy problems in linear on state problem (1), (2) (the matrix function $A(x, t)$, vector-valued functions $b(x, t)$ and $a(x, t)$, and functions $\varphi(x)$, $d(x, t)$ are linear with respect to x).

Pair $((x(t, u^0), \psi(t, u^0))$, as $t \in T$ satisfies the improvement boundary value problem for control u^0 satisfied to Pontryagin's maximum principle.

Developed procedure allows to improve controls, satisfied to Pontryagin's maximum principle, on account of non-unique solution of a improvement boundary value problem.

Define improvement condition in control space that is equivalent to the nonlocal boundary-value improvement problem (6), (7) in state space.

Let $(x^\alpha(t), p^\alpha(t))$, $t \in T$ be a solution of the boundary-value problem (6), (7) in state space. Then the admissible control $v^\alpha(t) = u^\alpha(p^\alpha(t), x^\alpha(t), t)$, $t \in T$ satisfies the condition

$$v(t) = u^\alpha(p(t, u^0, v), x(t, v), t), \quad t \in T \tag{9}$$

in control space. On the contrary, if $v^\alpha(t)$, $t \in T$ is an admissible control that is satisfying the relation (9), then pair $(x(t, v^\alpha), p(t, u^0, v^\alpha))$, as $t \in T$ satisfies the boundary value problem (6), (7). So, the boundary-value improvement problem (6), (7) in state space reduces to the condition (9) on the set of admissible controls V.

In the problem (1), (2), linear with respect to state, for solving the improvement control problem $u^0 \in V$ it is sufficient to solve two Cauchy problems in state space.

Note that here the condition (9) has the following form

$$v(t) = u^\alpha(\psi(t, u^0), x(t, v), t), \quad t \in T.$$

In the problem (1), (2), nonlinear with respect to state, for improvement u^0 it is possible to use method for solving the relation (9) on the set of admissible controls.

The difficulties in realization of the condition (9) are analogous to difficulties in solving the corresponding boundary-value improvement problem. In common case these difficulties are connected with features of projective mapping u^α.

4 Projective Perturbation Method

For given control $u^0 \in V$ and fixed $\alpha > 0$ let us represent the improvement condition (9) in control space in the form

$$v(t) = P_U(u^0(t) + \alpha H_1(p(t, u^0, v), x(t, v), t)), \ t \in T. \tag{10}$$

Let us consider a projection parameter $\alpha > 0$ as a perturbation parameter, and call the condition (9) perturbed. The unperturbed condition is obtained from the perturbed one (10) as $\alpha = 0$, and has the obvious solution $v(t) = u^0(t)$, $t \in T$.

The iterative process for solving the perturbed relation (10) has the form

$$v^{k+1}(t) = P_U(u^0(t) + \alpha H_1(p(t, u^0, v^k), x(t, v^k), t)), \quad t \in T, \ k \geq 0. \tag{11}$$

An initial approximation $v^0 \in V$ as $k = 0$ is prescribed on the initial (zero) iteration.

Let us formulate conditions for process convergence (11) on the set $V = \{v \in C(T) : v(t) \in U, t \in T\}$. For this purpose we will describe the perturbed problem (10) with respect to parameter $\alpha > 0$ and the process (11) for solving this problem in the operator form

$$v = G^\alpha(v), \quad v \in V, \tag{12}$$

$$v^{k+1} = G^\alpha(v^k), \quad k \geq 0, \tag{13}$$

where operator G^α is a superposition of three operators.

The first operator P is defined on the basis of the adjoint system using the relation

$$P(v) = p, \quad v \in V, \quad p(t) = p(t, u^0, v), \quad t \in T.$$

The second operator X is defined by solution $x(t, v)$, $t \in T$ for the phase system

$$X(v) = x, \quad v \in V, \quad x(t) = x(t, v), \quad t \in T.$$

The third operator V^α has the following form

$$V^\alpha(p, x) = v^\alpha, \quad p \in C(T), \quad x \in C(T), \quad v^\alpha(t) = u^\alpha(p(t), x(t), t), \quad t \in T.$$

Finally, G^α is represented in the form of composition

$$G^\alpha(v) = V^\alpha(P(v), X(v)), \quad v \in V.$$

In view of properties of the projection operator P_U the mapping G^α, $\alpha > 0$ is a single-valued.

The unperturbed problem

$$v = G^0(v), \quad v \in V$$

is defined by the operator $G^0 : v \to u^0$, $v \in V$. Therefore, u^0 is a unique solution of the unperturbed problem. In this case G^0 is obtained from G^α, if assume $\alpha = 0$.

The iterative process (13) has a form of standard simple iteration method for solving the operator equation (12). Conditions of convergence of simple iteration method can be defined on the basis of the known principle of contraction mappings. Let us formulate an analog of the known theorem [4].

Consider the operator $G : V \to V$, acting on the set V in completed normalized space of functions, that are defined on the set T with values in the compact set $U \subset R^m$, with the norm$\|\cdot\|_V$.

For solving the operator equation

$$v = G(v), \quad v \in V \tag{14}$$

the simple iteration method is considered

$$v^{k+1} = G(v^k), \quad k \geq 0. \tag{15}$$

Theorem 1. *Let the operator G satisfies the Lipschitz condition in the ball $B(v_0, l) = \{v \in V : \|v - v_0\|_V \leq l, v_0 \in V, l > 0\}$ with a constant $0 < M = M(v_0, l) < 1$*

$$\|G(v) - G(u)\|_V \leq M \|v - u\|_V, \; v \in B(v_0, l), \; u \in B(v_0, l),$$

moreover, the following condition is fulfilled

$$\|G(v_0) - v_0\|_V \leq (1 - M)l.$$

Then the equation (14) has a unique solution $\bar{v} \in B(v_0, l)$ and the simple iteration method (15) converges to $\bar{v}$ in the norm $\|\cdot\|_V$ at any initial approximation $v^0 \in B(v_0, l)$. The following estimate is correct for method error

$$\left\|v^k - \bar{v}\right\|_V \leq M^k \left\|v^0 - \bar{v}\right\|_V, \quad k \geq 0.$$

The theorem proof is similar to the proof illustrated in the work [4].

Assume that the family of phase trajectories is bounded on the set V:

$$x(t, v) \in X, \, t \in T, \, v \in V,$$

where $X \subset R^n$ is a convex compact set.

Note that the sufficient boundedness condition is fulfillment of the known estimate [1], [2]

$$\|f(x, u, t)\| \leq C(\|x\| + 1), \; x \in R^n, \; u \in U, \; t \in T.$$

Since the function $f(x,u,t) = A(x,t)u + b(x,t)$ is quadratic with respect to x, the Lipschitz condition is fulfilled

$$\|f(x,u,t) - f(y,u,t))\| \leq M_1 \|x - y\|, \ x \in X, \ y \in X, \ u \in U, \quad t \in T.$$

where $M_1 = const > 0$.

Hence, using the Gronwall-Bellman lemma [1], [2] it is possible to show that the operator X satisfies the Lipschitz condition

$$\|X(u) - X(v)\|_C \leq M_2 \|u - v\|_C, \ u \in V, \ v \in V,$$

where $M_2 = const > 0$.

Note that by virtue of linearity of the adjoint system the boundedness condition for the family of adjoint trajectories is fulfilled on the basis of the sufficient condition

$$p(t, u^0, u) \in P, \quad t \in T, \quad u \in V,$$

where $P \subset R^n$ is a convex compact set.

Hence, taking into account the Lipschitz condition for operator X, it is easy to obtain the Lipschitz estimate for the operator P

$$\|P(v) - P(u)\|_C \leq M_3 \|v - u\|_C, \ v \in V, \ u \in V,$$

where $M_3 = const > 0$.

On the basis of the Lipschitz condition for the projection operator P_U we obtain

$$\begin{aligned}\|u^\alpha(p(t),x(t),t) - u^\alpha(q(t),y(t),t)\| =& \alpha \|H_1(p(t),x(t),t) - H_1(q(t),y(t),t)\| \\ & \leq \alpha M_4(\|p-q\|_C + \|x-y\|_C),\end{aligned}$$

where $t \in T$, $p,x,q,y \in C(T)$, and $M_4 = const > 0$. Therefore,

$$\|V^\alpha(p,x) - V^\alpha(q,y)\|_C \leq \alpha M_4(\|p-q\|_C + \|x-y\|_C), \quad p,x,q,y \in C(T).$$

So, the operator V^α satisfies the Lipschitz condition with a constant, proportional to parameter $\alpha > 0$. From estimations it follows that the operator G^α satisfies the Lipschitz condition with a constant, proportional to $\alpha > 0$

$$\|G^\alpha(v) - G^\alpha(u)\|_C \leq \alpha(M_2 + M_3)M_4 \|v - u\|_C, \quad v \in V, u \in V.$$

On the whole, by Theorem 1, the iterative process (13) at small $\alpha > 0$ converges to a unique solution of the perturbed problem (14) for any initial approximation $v^0 \in V$.

So, the following convergence theorem is proved.

Theorem 2. *Let the family of phase trajectories in the problem linear with respect to control and quadratic with respect to state* (1), (2) *with the convex compact set* $U \subset R^m$ *be bounded:*

$$x(t,u) \in X,\ t \in T,\ u \in V,$$

where $X \subset R^n$ *is a convex compact set.*

Then for a sufficient small projection parameter $\alpha > 0$

1) the problem (14) *has a unique solution* $\bar{v}^\alpha \in V$*;*

2) the iterative process (15) *converges in the norm* $\|\cdot\|_C$ *to a solution* $\bar{v}^\alpha$ *for any initial approximation* $v^0 \in V$.

Note that under conditions of theorem 2 the solution of the perturbed problem (10) for control $u^0 \in V$, satisfying the maximum principle, coincides with u^0 since its uniqueness.

For initial approximation of iterative processes (11) in solving the perturbed problem (10) for control $u^0 \in V$, that is not satisfying the maximum principle, it is possible to choose the initial approximation $v^0 = u^0$. In this case for sufficiently small $\alpha > 0$, according to theorem 2 and the improvement estimate (8), the strict improvement of control u^0 of iterative processes is guaranteed.

Note that in projective perturbation method the control $u^0 \in V$ is being improved by solving the perturbed problem (10) for any perturbation parameter $\alpha > 0$.