Some Non-Cooperative Games on Pasture Surfaces

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Abstract

In this paper we consider basic elements of so-called pasture territory and some related extremal[1,2,3] and game problems. This work may be considered as continuation of the paper [1] where the extremal problems for single herdsman are investigated. We describe the pasture surface as a graph of a piecewise smooth and continuous function $f(x, y)$ defined on a closed, connected domain of a plane. Considering extremal and non-cooperative game problems are related with finding the optimal location for the nomadic residences, when the exploiting pasture territory for herdsman has grass mass as much as possible[1,2].

Keywords: Pasture territory, Herbage density, piecewise smoothness, non-negative measure, watering-place, closure of a set, upper semi-continuity, convexity.

1 Introduction

The world civilization is divided into two forms: settled and nomadic. While the settled civilization is well studied and modeled mathematically, the study of the nomadic civilization is practically ignored and less. Therefore, our work may be regarded as new in mathematical modeling. The nomadic civilization is closely connected with the nature, and ecological and economical problems of nomads are regulated simultaneously.

Mongolia is one of the few countries where the nomadic civilization still exists in classical form. Fifty percent of the population is involved somehow in stock nomadic breeding. Since Mongolian has extreme climate, it is very important for nomads to determine optimal choices for roaming places, i.e., the location for the nomadic residence depending on the seasons. During the last 60 years, the livestock sector of Mongolia past two historical periods: totalitarian period, in which all problems are solved in government interest and transitional period of privatization, in which the nomads solve any problems their own way.

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In this section we define basic elements of the pasture surface. Let $K \subseteq \Re^2$ be a closure of an open and connected set with a piecewise smooth boundary. Suppose that K consists of a union of a finite number of domains K_i with piecewise smooth boundaries. Then the pasture surface is defined as a graph of a continuous function $f : K \to R$ such that $f(x, y)$ is twice differentiable on the interior of K_i for any i.

We define the watering place for the herd as a closure of a set $W \subseteq f(K)$ with an empty interior. That means the pasture surface does not contain the interior of the water resource[1,2,3].

We denote the closed set $Q \subseteq f(K)$ as possible locations for the nomadic residence.

Theorem 1. *Between any two points in $f(K)$, there exists a curve of minimal length (minimal curve[5]) in $f(K)$ connecting them.*

Proof. Suppose $O_1, O_2 \in f(K)$, $O_1 \neq O_2$. From the connectedness of K, it follows that the points $f^{-1}(O_1)$ and $f^{-1}(O_2)$ can be connected by a rectifiable planar curve l. Then $f(l)$ is also a rectifiable surface curve with a length d. Let us construct a planar disk $B(f^{-1}(O_1), d) := \{z \in \Re^2 | \ \|z - f^{-1}(O_1)\|_2 \leq d\}$ with a center $f^{-1}(O_1)$ and a radius d. Then the graph $f(B(f^{-1}(O_1), d) \cap K)$ is a complete metric space with the surface metric. This space, evidently, contains the curve $f(l)$ and the point O_2. Hence, by the theorem 3 (P.112) of [4], there exists a minimal surface curve connecting O_1 and O_2. □

If the nomadic residence is located at the point $O \in Q$, we define the maximal possible exploiting area $A_r(O, \bar{W}) \subseteq f(K)$ as the union of all points $M \in f(K)$ such that there exists a loop $l \subseteq f(K)$ of length no more than $2r$ passing through the points M, O and some point $N \in \overline{W}$. This means, for a day, while grazing and watering one's livestock, the herdsman must pass the distance no more than $2r$. The $r > 0$ is called the radius of grazing. It is clear that $A_r(O, \bar{W})$ is a connected compact set[1,2].

Pasture surface $f(K)$ is a complete metric space, where the distance $\rho_1(M, N)$ for the points $M, N \in f(K)$ is equal to the length of a minimal curve connecting them. This metric ρ_1 is called the surface metric. Any minimal curve consists of possible pieces of the boundary $\partial f(K)$ and some geodesics.

Surface ellipse $E_r(O_1, O_2)$ with focuses $O_1, O_2 \in f(K)$ is a compact set satisfying

$$\rho_1(O_1, M) + \rho_1(O_1, O_2) + \rho_1(M, O_2) \leq 2r, \ \forall M \in E_r(O_1, O_2).$$

Each shoot of the boundary $\partial E_r(O_1, O_2)$ is a closed curve.

When $\rho_1(O_1, O_2) = r$, $int\ E_r(O_1, O_2) = \emptyset$.

When $\rho_1(O_1, O_2) < r$, $int\ E_r(O_1, O_2) \neq \emptyset$.

We denote by $W_r(O)$ the subset of $\overline{W}$ such that

$$W_r(O) := \{N \in \overline{W}| \ \ \rho_1(O,N) \le r\}.$$

Assume that $\rho_1(O,N) < r$ for any $N \in W_r(O)$. Then the next theorem holds.

Theorem 2. *The boundary $\partial(\overline{int\ A_r(O,\bar{W})})$ is a union of a finite number of closed, rectifiable curves.*

Proof. Since

$$A_r(O,\bar{W}) = \bigcup_{N \in W_r(O)} E_r(O,N),$$

the boundary $\xi(0) = \partial A_r(O,\bar{W})$ consists of $\partial(\overline{int\ A_r(O,\bar{W})})$ and some possible shoots. It is clear that $\overline{intA_r(O,\bar{W})}$ is a union of a family of ellipses $E_r(O,N),\ N \in W_r(O)$, where $\overline{intE_r(O,N)} = E_r(O,N)$. Since $\overline{intA_r(O,\bar{W})}$ is a compact set, we can choose some ellipses $E_r(O,N_1),...,E_r(O,N_k)$ covering $\overline{intA_r(O,\bar{W})}$ in union. As each $\partial E_r(O,N_i), i = \overline{1,k}$ is a union of a finite number of closed and rectifiable curves, $\partial(\overline{int\ A_r(O,\bar{W})})$ also is a union of a finite number of closed and rectifiable curves. □

Corollary 1. *When there exists only a finite number of points $N_i \in W_r(O)$ satisfying $\rho_1(O,N_i) = r$ and total length of the shoots of $\partial A_r(O,\bar{W})$ is finite, the boundary $\xi(O) = \partial A_r(O,\bar{W})$ has a finite length.*

Herbage density is a non-negative measure $\mu(K)$ such that for any compact set $M \subseteq K$,

$$\mu(M) < \infty$$

and the charge $Z(A)$ generated by bounded function $g(x,y) = \sqrt{1+f_x^2+f_y^2}$:

$$Z(A) = \int\limits_A \sqrt{1+f_x^2+f_y^2}\ d\mu$$

Proof. Consider the sequence $\xi_n = \bigcup_{i=1}^{k} \xi_n^i$, where each sequence ξ_n^i converges to ξ^i with respect to the above metric in Ξ. If we denote

$$S(\eta_n) = \sup_{i \geq n} S(\xi_i) \quad \text{with} \quad \inf_{\xi_j \in \bigcup_{i \geq n} \xi_i} \rho_{max}(\eta_n, \xi_j) = 0,$$

$$\rho_{max}(\xi_1, \xi_2) = \max_{1 \leq i \leq k} \rho(\xi_1^i, \xi_2^i),$$

then we have

$$S(\eta_n) = S(\xi) + \int\limits_{\Pi_{\eta_n} \setminus (\Pi_{\eta_n} \cap \Pi_\xi)} \sqrt{1 + f_x^2 + f_y^2} d\mu - \int\limits_{\Pi_\xi \setminus (\Pi_{\xi_0} \cap \Pi_\xi)} \sqrt{1 + f_x^2 + f_y^2} d\mu \ .$$

When n goes to infinity the first integral tends to zero, but the second integral tends to -$\mu(\xi_0)$, where ξ_0 is a piece of the curve ξ. Therefore, $S(\xi) \geq \overline{\lim_{n\to\infty}} S(\xi_n)$ and the lemma is proved. □

For any $O \in f(K)$, we introduce a notation $O^p = f^{-1}(O)$.

Theorem 3. *Function $W(O)$ (respectively $W(O^p)$) given in* (1) *is upper semi-continuous on $f(K)$ (respectively K).*

Proof. Let $O_n^p \to O^p$ be a sequence in K. Then the sequence $O_n = f(O_n^p)$ also tends to O in $f(K)$. Suppose that ξ_n is a boundary of $A_r(O_n, \bar{W})$ consisting of some k closed continuous curves. It is clear that $O_n \to O$ ($O_n^p \to O^p$) implies $\xi_n \to \xi$. By previous lemma, the function $W(O)$ ($W(O^p)$) is also upper semi-continuous. □

Corollary 2. *If $Q(f^{-1}(Q))$ is compact, then the problem* (1) *has a solution on $Q(f^{-1}(Q))$.*

Now we consider the problem of locating optimally residences for L herdsmen. The traditional pasture selection method of mongols, indeed, is very much alike non-cooperative games.

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We will consider the case when nomads are using the pasture territory area $f(K)$. For simplicity, assume that every i-th stock nomadic-breeding has finite points $O^i_{s_i}$, $1 \leq s_i \leq k_i\}$ for residence (ger) location. If his residence is located at point $Q^i_{s_i}$, maximal pasture territory is $A_r(Q^i_{s_i}, \bar{W}) \subseteq f(K)$. We will consider all systems of $\{1_{s_1}, \ldots, L_{s_L}\}$, where $1 \leq s_i \leq k_i$. The number of family $\{s_1, \ldots, s_L\}$ is equal to $k_1 \times \ldots \times k_L$. For fixed $s_1, \ldots, s_L$ we define vector $q = (q_1, \ldots, q_L)$, where q_i is equal to either 0 or 1. If $q_i = 1$, then the pasture territory $A_r(O^i_{s_i}, \bar{W})$ consists of intersection subsets, the number of them is equal to 2^{L-1}. The number of elements of intersection

$$\bigcap_{j:q_j=1} A_r(O^j_{s_j}, \bar{W}) \;\; is \;\; l_i(q) = \sum_{j=1}^{L} q_j.$$

Suppose that every herdsman has N number of animals. Assume that i-th herdsman has mixed strategy

$$x_i = \left(x^i_1, \ldots, x^i_{k_i}\right) \Big| \sum_{p=1}^{k_i} x^i_p = 1, \quad x^i_p \geq 0,$$

where x^i_p is the probability of choosing O^i_p, $\; p = 1, \ldots, k_i$. Then the usable weighted-mean pasture area for i-th herdsman, i.e. payoff of i-th player is

$$W_i\left(x_1, \ldots, x_L\right) = \sum_{(s_1,\ldots,s_L)} \prod_{i=1}^{L} x^i_{s_i} \sum_{(q_1,\ldots,q_L)} \frac{1}{l_i(q)} \int\limits_{\bigcap\limits_{j:q_j=1} A_r(O^j_{s_j}, \bar{W})} d\mu. \qquad (2)$$

Therefore, the Nash equilibrium $(x^*_1, \ldots, x^*_L)$ is defined as

$$W_i(x^*_1, \ldots, x^*_L) \geq W_i(x^*_1, \ldots, x_i, \ldots, x^*_L), \quad x_i \in X_i$$

for any $i \in \overline{1, L}$, where X_i set of mixed strategies of i-th herdsman.

Theorem 4. *Non-cooperative game problem defined by* (2) *always has a Nash equilibrium.*

Proof. Each function $W_i(x_1, \dots, x_N)$ is continuous in x_i for fixed $L-1$ arguments of $x_j,\ j \neq i$. And X_i is a $(k_i - 1)$ dimensional simplex,therefore, it is convex and compact. Hence the theorem is proved[8]. $\square$

3 Some illustrative examples

In this section we assume that f is a linear function. For simplicity, we assume that $f(K) = R^2$, W consists from a finite number of wells and the grass cover is uniformly distributed. Also suppose that $r = 1$ and the distance between ger and the nearest well is less than 1. We use mixed strategies and Nash equilibrium[8] for the optimal mixed strategies of considering games.

A. Two herdsmen, two wells. Assume that the distance between two fixed wells is equal to $2a (0 < a < 1)$ and every herdsman has right to place his ger near any one of the wells. Suppose that the probability of ger location for first herdsman near first well is $0 \leq x \leq 1$, but the probability of ger location for second herdsman near second well is $0 \leq y \leq 1$. Then we have a bimatrix game and we denote the pure strategy for ger locating and no ger locating by 1 and 0, respectively. Then the next 4 game situations of pure strategies hold: $(1,1), (1,0), (0,1), (0,0)$, The goal of every herdsman is to maximize his weighted-mean pasture area as much as possible. This is symmetric game and so it is sufficient to consider only the payoff function of the first player.

$$\begin{aligned} F_1(x,y) &= C(a)xy + \frac{\pi}{2}x(1-y) + \frac{\pi}{2}(1-x)y + [\pi - C(a)(1-x)(1-y)] \\ &= (C(a) - \frac{\pi}{2})(2y-1)x - (\frac{3\pi}{2} - C(a))y + C(a), \end{aligned}$$

where $C(a) = \pi - (\arccos a - a\sqrt{1-a^2})$. For the payoff function of second player $F_2(x,y)$, we need to interchange x and y in above formula. Finding the maximum value of $F_1(x,y)$ in x and the maximum value of $F_2(x,y)$ in y, we find the necessary conditions for Nash equilibriums:

$$x^* = \begin{cases} 1, & y^* > \frac{1}{2}, \\ [0,1], & y^* = \frac{1}{2}, \\ 0, & y^* < \frac{1}{2}, \end{cases} \quad y^* = \begin{cases} 1, & x^* > \frac{1}{2}, \\ [0,1], & x^* = \frac{1}{2}, \\ 0, & x^* < \frac{1}{2}. \end{cases}$$

Hence the Nash equilibriums are $(1,1),(0,0),(\frac{1}{2},\frac{1}{2})$.

B. Two herdsmen and one well. let the distances from two fixed residence places to well satisfy $0 < a < 1$. Assume that the first player has two pure strategies: to place at first place or near well and second player has two pure strategies: to place at second place or near well. Suppose that the probability of the first herdsman's ger location being at first place is $0 \le x \le 1$, but the probability of second herdsman's ger location being at second place is $0 \le y \le 1$. Since the ellipse area is $\pi\sqrt{1-a}$, the playoff for the first player is defined as follows:

$$\begin{aligned} F_1(x,y) =& (\pi\sqrt{1-a} - S(a))xy + \frac{1}{2}\pi\sqrt{1-a}(1-y)x + \pi(1-\frac{1}{2}\sqrt{1-a})(1-x)y \\ &+ \frac{1}{2}\pi(1-x)(1-y) \\ =& [(\pi\sqrt{1-a} - \frac{1}{2}S(a) - \frac{\pi}{2})y + \frac{1}{2}\pi\sqrt{1-a} - \frac{1}{2}\pi]x + \frac{1}{2}\pi + \frac{\pi}{2}(1-\sqrt{1-a})y. \end{aligned}$$

Where $S(a)-$ is the area of intersection of two ellipses. By interchanging x and y, we obtain the formula for $F_2(x,y)$. Since

$$\pi\sqrt{1-a} - \frac{1}{2}S(a) - \frac{\pi}{2} < 0$$

and

$$\frac{1}{2}\pi\sqrt{1-a} - \frac{1}{2}\pi < 0,$$

$(0,0)$ is the unique Nash equilibrium of this game.

C. Three herdsmen, tree wells. Suppose that vertices of equilateral triangle are located at wells and $BA = AC = BC = a < 2$. Assume that the first player has two pure strategies: to place at points A and B , the second player has two pure strategies: to place at points B and C and the third player has two pure strategies:

to place at points C and A . Suppose that the probability of first herdsman's ger location being at A is $0 \le x \le 1$, but the probability of second herdsman's ger location being at B is $0 \le y \le 1$ and the probability of third herdsman's ger location being at C is $0 \le z \le 1$. Let $S(a)$ be the area of intersection of three circles with centers at A, B and C of unit radius. We have

$$\begin{aligned} F_1(x,y,z) = & \left(\frac{1}{3}S(a) + C(a)\right) xyz + \left(\frac{\pi}{3} + \frac{1}{6}C(a)\right) xy(1-z) \\ & + \left(\frac{\pi}{3} + \frac{2}{3}C(a)\right) x(1-y)z \\ & + \left(\frac{\pi}{3} + \frac{1}{6}C(a)\right) (1-x)yz + \frac{1}{6}C(a)(1-x)y(1-z) \\ & + \left(\frac{\pi}{3} + \frac{1}{6}C(a)\right) x(1-y)(1-z) + \left(\frac{\pi}{3} + \frac{2}{3}C(a)\right) (1-x)(1-y)z \\ & + \left(\frac{1}{3}S(a) + C(a)\right) (1-x)(1-y)(1-z) \\ = & \left(-\frac{\pi}{3} + \frac{1}{3}S(a) + \frac{5}{6}C(a)\right) (y+z-1) + Q(y,z). \end{aligned}$$

By interchanging x and y in this formula we obtain $F_2(x,y,z)$, and by interchanging x and z we obtain $F_3(x,y,z)$.

Finding the maximum values of $F_1(x,y,z)$,$F_2(x,y,z)$,$F_3(x,y,z)$ in arguments x, y, z, respectively we will find the necessary conditions for all Nash equilibriums. Since the inequality

$-\frac{\pi}{3} + \frac{1}{3}S(a) + \frac{5}{6}C(a) > 0$ is always satisfied, for a Nash equilibrium (x^*, y^*, z^*) we have:

$$x^* = \begin{cases} 1, & y^* + z^* > 1, \\ [0,1], & y^* + z^* = 1, \\ 0, & y^* + z^* < 1. \end{cases} \quad y^* = \begin{cases} 1, & x^* + z^* > 1, \\ [0,1], & x^* + z^* = 1, \\ 0, & x^* + z^* < 1. \end{cases} \quad z^* = \begin{cases} 1, & x^* + y^* > 1, \\ [0,1], & x^* + y^* = 1, \\ 0, & x^* + y^* < 1. \end{cases}$$

Hence we have three Nash equilibriums: $(1,1,1)$, $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(0,0,0)$.

4 Addition

In this section we will consider some problems linked with form of utility function, set of residence location and situation in which a herdsman may have more than one residences. Assume that we have a pasture territory $M \in f(K)$ and the number of cattle grazing on this territory is N. Then the grass biomass per animal is equal to

$$G = \frac{1}{N} \int\limits_{M} d\mu.$$

We assume that the utility function $W(G)$ is:

$$W(G) \;=\; \begin{cases} \int\limits_{M} d\mu, & if \quad G \;\leq\; G_{cr}, \\ NG_{cr}, & if \quad G \;>\; Gcr, \end{cases}$$

where $G_{cr} \;>\; 0$ fixed number. Consequently, the formula (2) of payoff function is for $G \leq G_{cr}$, I.,e. for sufficiently large N.

In fact, when the number of herdsmen $L > 1$ and livestock number of $i-$th herdsman is equal to $N_i, 1 \leq i \leq L$, and in place of (2), the next formula for the payoff function holds.

$$W_i\left(x_1, \ldots, x_L\right) = \sum_{(s_1,\ldots,s_L)} \prod_{i=1}^{L} x^i_{s_i} \sum_{(q_1,\ldots,q_L)} F_i(q),$$

where

$$F_i(q) = \begin{cases} N_i \bar{F}_i(q), & if \quad \bar{F}_i(q) \leq G_{cr}, \\ N_i G_{cr}, & if \quad \bar{F}_i(q) > Gcr \end{cases} .$$

Where is used the next designation:

$$\bar{F}_i(q) = \frac{1}{\sum\limits_{j=1}^{L} q_j N_j} \int\limits_{\bigcap\limits_{j:q_j=1} A_r(O^j_{s_j}, \bar{W})} d\mu$$

In general case a herdsman may have any closed subset $Q \subseteq f(K)$ for his residence locations.

Now we will consider again the case $L = 1$. Let a herdsman have p residences and he must choose location points $O_i \in Q,\ i \in \overline{1,p}$ and allocate the livestock number:

$$N = \sum_{1}^{p} N_i$$

as well as. For example, we will consider next problems, where $p = 2$.

Let M be a pasture region which is a convex and closed polygon in a plane and $M = \bar{W} = Q$, $d_M \geq 2r$, where d_M is the measure of length of a segment in M with maximal length. In addition assume that,

$$\frac{2\pi r_M^2}{N} \leq G_{cr} \tag{3}$$

and the grass cover is uniformly distributed on M. A herdsman's goal is choosing $O_1 \in Q$, $O_2 \in Q, N_1(N_2 = N - N_1.)$

Problem 1. *To find O_1^*, O_2^*, N_1^* such that*

$$S(B(O_1^*, r) \cap M) + S(B(O_2^*, r) \cap M) = \max_{O_1, O_2 \in M, N_1 \in (0,N)} \sum_{i=1}^{2} S(B(O_i, r) \cap M),$$

where $intB(O_1, r) \cap intB(O_2, r) = \emptyset,\ O_1,\ O_2 \in M$ and $S(D)$ is the area of domain D.

Problem 2. *To find O_1^*, O_2^*, N_1^* such that*

$$\pi(r_1^*)^2 + \pi(r_2^*)^2 \geq \pi(r_1)^2 + \pi(r_2)^2,$$

where int $B(O_1, r_1) \cap$ int $B(O_2, r_2) = \emptyset, B(O_i, r_i) \subseteq M, i = 1, 2.$

Lemma 2. *Problems 1 and 2 always have a solution.*

Proof. The condition (3) guarantees that the utility function for the herdsman over $B(O_i, r) \cap M$ for Problem 1 is

$$\int\limits_{B(O_i,r)\cap M} d\mu$$

and the utility function over $B(O_i, r_i)$ for Problem 2 is

$$\int\limits_{B(O_i,r_i)} d\mu = \pi r_i^2.$$

In fact, if for a distribution $N = \sum_{i=1}^{2} N_i$ of livestock numbers, we have

$$\frac{1}{N_i} \int\limits_{B(O_i,r)\cap M} d\mu > G_{cr} \quad and \quad \frac{1}{N_i} \int\limits_{B(O_i,r)} d\mu > G_{cr},$$

then by increasing N_i(therefore, simultaneously increasing corresponding sum of utility functions), we get

$$\frac{1}{N_i} \int\limits_{B(O_i,r)\cap M} d\mu \le G_{cr} \quad and \quad \frac{1}{N_i} \int\limits_{B(O_i,r)} d\mu \le G_{cr},$$

respectively. □

In future, we assume that N_i is fixed. Now we describe the mathematical formalization of above two problems. First, we will consider the Problem 1. Assume that the convex polygon M is constructed as an intersection of nonnegative sides of lines: $a_i x + b_i y + c_i = 0, i = 1, \ldots, n$. Let (x_1, y_1) and (x_2, y_2) be the coordinates of points O_1 and O_2, respectively. Let $A_i,\ i = 1, \ldots, n$ be the vertexes of polygon and $s_i = A_i A_{i+1},\ i = 1, \ldots, n+1$, where $A_{n+1} = A_1$. We denote by $r_{j,i},\ j = 1, 2$ the distance from point O_j to side s_i. Then the areas $S(B(O_j, r) \cap M), j = 1, 2$ for Problem 1 are equal to

$$\frac{1}{2} \sum_{p=1}^{m_j} r_{j,k_p} \cdot \left(\sqrt{(\min(r, O_j A_{k_p}))^2 - r_{j,k_p}^2} + \sqrt{(\min(r, O_j A_{k_{p+1}}))^2 - r_{j,k_p}^2} \right) +$$