Semilocal convergence with R-order four iteration

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Abstract

In this paper we prove a semilocal convergence theorem for an R-order four iteration free from second derivative under conditions similar to those of the Newton-Kantorovich theorem and give a priori error bound. Numerical comparisons are made to show the performance of the presented method and its modifications.

1 Introduction

In recent years have been appeared many higher order methods [1-4] for solving nonlinear equations. They can be used in problems, where a quick convergence is required, such as stiff systems [3,4]. We consider the following fourth order iteration [8,5]:

$$y_n = x_n - \Gamma_n F(x_n), \quad x_{n+1} = y_n - F'(y_n)^{-1}F(y_n), \quad \Gamma_n = F'(x_n)^{-1}, \tag{1}$$

for solving nonlinear equation $F(x) = 0$. Here we assume that X and Y are Banach spaces and $F : \Omega \subseteq X \to Y$ is a nonlinear twice differentiable Frechet operator defined on a convex nonempty domain Ω. Let us also assume that $F'(x_0)^{-1} \in \mathscr{L}(Y, X)$

exists for some $x_0 \in \Omega$, where $\mathscr{L}(Y, X)$ is the set of bounded linear operators from Y into X. Moreover we suppose that

(c_1) $\|\Gamma_0\| \leq \beta$,

(c_2) $\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta$,

(c_3) $\|F''(x)\| \leq M, \qquad x \in \Omega.$

In this paper, we smoothen the conditions imposed on operator F. Namely, we prove a semilocal convergence theorem for (1) without assumption of Lipschitz continuity of the second derivative and obtain error bounds by using a technique consisting of a new system of recurrence relations [2,3].

2 Convergence study

We denote

$$a_0 = M\beta\eta, \tag{2}$$

$$f(x) = \frac{2(1-x)}{2(1-x)^2 - x^2}, \quad g(x) = \frac{x^3}{8(1-x)^2} \tag{3}$$

and define a sequence

$$a_{n+1} = f(a_n)^2 g(a_n) a_n, \qquad n = 0, 1, \ldots \tag{4}$$

Firstly, technical lemmas, the proofs of which are trivial, are provided [2,3].

Lemma 1. *Let f and g be two real functions given by (3). Then*

(i) They are increasing and $f(x) > 1$ for $x \in (0, 1/2)$.

(ii) $f(\gamma x) < f(x)$ and $g(\gamma x) \leq \gamma^3 g(x)$ for $x \in (0, 1/2)$ and $\gamma \in (0, 1)$.

Lemma 2. *Let* $0 < a_0 < 1/2$. *Then* $f^2(a_0)g(a_0) < 1$ *and the sequence* $\{a_n\}$ *is decreasing.*

Lemma 3. *Let us suppose that the hypothesis of lemma 2 is satisfied and define* $\gamma = a_1/a_0$. *Then*

(i) $\gamma = f(a_0)^2 g(a_0) \in (0,1)$

(ii) $a_n \le \gamma^{4^n} a_{n-1} \le \gamma^{\frac{4^n-1}{3}} a_0$

(iii) $f(a_n)g(a_n) < \Delta\gamma^{4^n}, \qquad \Delta = \frac{1}{f(a_0)} < 1.$

Notice that

$$M\|\Gamma_0\| \cdot \|\Gamma_0 F(x_0)\| \le a_0, \qquad \|y_0 - x_0\| \le \eta < R\eta$$

and

$$\begin{aligned} \|x_1 - x_0\| &\le \left(1 + \frac{a_0}{2(1-a_0)}\right)\|y_0 - x_0\| \\ &< \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{1}{1-\gamma\Delta}\eta = R\eta, \end{aligned}$$

where $R = \left(1 + \frac{a_0}{2(1+a_0)}\right)\frac{1}{1-\gamma\Delta}$. This means that $y_0, x_1 \in B(x_0, R\eta) = \{x \in X \| \|x - x_0\| < R\eta\}$. In these conditions we prove, for $n \ge 1$, the following statements:

(I_n) $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \le f(a_{n-1})\|\Gamma_{n-1}\|$,

(II_n) $\|\Gamma_n F(x_n)\| \le f(a_{n-1})g(a_{n-1})\|\Gamma_{n-1}F(x_{n-1})\|$,

(III_n) $M\|\Gamma_n\| \cdot \|\Gamma_n F(x_n)\| \le a_n$,

(IV_n) $\|x_{n+1} - x_n\| \le \left(1 + \frac{a_n}{2(1-a_n)}\right)\|y_n - x_n\|$,

(V_n) $y_n, x_{n+1} \in B(x_0, R\eta)$.

Assuming

$$\left(1+\frac{a_0}{2(1-a_0)}\right)a_0<1, \qquad x_1\in\Omega$$

we have

$$\|I-\Gamma_0F'(x_1)\|\le\|\Gamma_0\|\cdot\|F'(x_0)-F'(x_1)\|\le M\|\Gamma_0\|\cdot\|x_1-x_0\|\le\left(1+\frac{a_0}{2(1-a_0)}\right)a_0<1.$$

Then, by the Banach lemma, Γ_1 is defined and

$$\|\Gamma_1\|\le\frac{\|\Gamma_0\|}{1-\|\Gamma_0\|\cdot\|F'(x_0)-F'(x_1)\|}\le\frac{1}{1-\left(1+\frac{a_0}{2(1-a_0)}\right)a_0}\|\Gamma_0\|=f(a_0)\|\Gamma_0\|.$$

On the other hand, if $x_n, x_{n-1}\in\Omega$ we obtain from Taylor's formula

$$F(x_{n+1})=\frac{F''(\eta_n)}{2}(x_{n+1}-y_n)^2,\quad \eta_n=\alpha y_n+(1-\alpha)x_{n+1},\quad \alpha\in(0,1). \tag{5}$$

Therefore using

$$x_{n+1}-y_n=-F'(y_n)^{-1}F(y_n)$$

and

$$F(y_n)=\frac{F''(\xi_n)}{2}(y_n-x_n)^2,\quad \xi_n=\theta x_n+(1-\theta)y_n,\quad \theta\in(0,1)$$

we get

$$F(x_{n+1})=\frac{F''(\eta_n)}{8}\left(F'(y_n)^{-1}\right)^2\left(F''(\xi_n)\right)^2(y_n-x_n)^4. \tag{6}$$

Analogously, we have

$$\begin{aligned}F'(y_n)&=F'(x_n)+F''(\bar{\xi_n})(y_n-x_n)=F'(x_n)\left[I+\Gamma_nF''(\bar{\xi_n})(y_n-x_n)\right],\\ \bar{\xi_n}&=wx_n+(1-w)y_n,\qquad w\in(0,1).\end{aligned}$$

Therefore, we obtain

$$\|F'(y_n)^{-1}\|\le\|(I+\Gamma_nF''(\bar{\xi_n})(y_n-x_n))^{-1}\|\cdot\|\Gamma_n\|\le\frac{1}{1-a_n}\|\Gamma_n\|. \tag{7}$$

Then, for $n = 0$, if $y_0 \in \Omega$, we have

$$\begin{aligned}
\|F'(y_0)^{-1}\| &\leq \frac{1}{1-a_0}\|\Gamma_0\|, \\
\|\Gamma_1 F(x_1)\| &\leq \|\Gamma_1\| \cdot \|F(x_1)\| \leq f(a_0)\|\Gamma_0\|^3 \frac{M^3}{8}\|y_0 - x_0\|^4 \\
&< f(a_0)g(a_0)\|\Gamma_0 F(x_0)\|
\end{aligned}$$

and (II_1) is true. To prove (III_1) and (IV_1), notice that

$$M\|\Gamma_1\| \cdot \|\Gamma_1 F(x_1)\| \leq M f(a_0)^2 g(a_0)\|\Gamma_0\| \cdot \|\Gamma_0 F(x_0)\| = a_1.$$

In addition, we easily deduce that

$$\begin{aligned}
\|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| \leq \left(f(a_0)g(a_0) + 1 + \frac{a_0}{2(1-a_0)}\right)\eta \\
&= \left(\gamma\Delta + 1 + \frac{a_0}{2(1-a_0)}\right)\eta < \left(1 + \frac{a_0}{2(1-a_0)}\right)(1+\gamma\Delta)\eta \\
&< \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{1}{1-\gamma\Delta}\eta = R\eta
\end{aligned}$$

and

$$\|x_2 - x_1\| \leq \left(1 + \frac{a_1}{2(1-a_1)}\right)\|\Gamma_1 F(x_1)\|.$$

Then, we have

$$\begin{aligned}
\|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \\
&\leq \left\{\left(1 + \frac{a_1}{2(1-a_1)}\right) f(a_0)g(a_0) + 1 + \frac{a_0}{2(1-a_0)}\right\}\eta \\
&\leq \left(1 + \frac{a_0}{2(1-a_0)}\right)(1+\Delta\gamma)\eta < \left(1 + \frac{a_0}{2(1-a_0)}\right)\frac{1}{1-\gamma\Delta}\eta = R\eta.
\end{aligned}$$

Therefore, $y_1, x_2 \in B(x_0, R\eta)$. Now following an inductive procedure and assuming

$$y_n, x_{n+1} \in \Omega \text{ and } a_n\left(1 + \frac{a_n}{2(1-a_n)}\right) < 1, \quad n \in \mathbf{N} \tag{8}$$

the items $(I_n) - (V_n)$ are proved. To establish the convergence of $\{x_n\}$ we only have to prove that it is a Cauchy sequence and that the above assumptions (8) are true.

We note that

$$\left(1+\frac{a_n}{2(1-a_n)}\right)\|\Gamma_n F(x_n)\| \le \left(1+\frac{a_n}{2(1-a_n)}\right) f(a_{n-1})g(a_{n-1})\|\Gamma_{n-1}F(x_{n-1})\|$$

and hence

$$\left(1+\frac{a_n}{2(1-a_n)}\right)\|\Gamma_n F(x_n)\| \le \left(1+\frac{a_0}{2(1-a_0)}\right)\|\Gamma_0 F(x_0)\| \prod_{k=0}^{n-1} f(a_k)g(a_k)$$

by induction. As a consequence of Lemma 3 it follows that

$$\prod_{k=0}^{n-1} f(a_k)g(a_k) \le \prod_{k=0}^{n-1} \gamma^{4^k}\Delta = \Delta^n \gamma^{\frac{4^n-1}{3}}.$$

Since $\Delta < 1$ and $\gamma < 1$, we deduce that $\prod_{k=0}^{n-1} f(a_k)g(a_k)$ converges to zero by letting $n \to \infty$. We can now state the following result on convergence for (1).

Theorem 1. *Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Frechet differentiable operator defined on a convex, nonempty domain Ω. Let us assume that $\Gamma_0 \in \mathscr{L}(Y, X)$ exists at some $x_0 \in \Omega$ and $(c_1) - (c_3)$ are satisfied. Suppose that $0 < a_0 < 1/2$. Then, if $\overline{B(x_0, R\eta)} = \{x \in X | \|x - x_0\| \le R\eta\} \subseteq \Omega$, the sequence $\{x_n\}$ defined in (1) and starting at x_0 has, at least, R-order four and converges to a solution x^* of the equation $F(x) = 0$. In that case, the solution x^* and the iterates y_n, x_n belong to $\overline{B(x_0, R\eta)}$, and x^* is the only solution of $F(x) = 0$ in $B(x_0, 2/M\beta - R\eta) \bigcap \Omega$. Furthermore, we have following error estimate:*

$$\|x^* - x^n\| \le \left(1+\frac{a_0}{2(1-a_0)}\gamma^{\frac{4^n-1}{3}}\right)\gamma^{\frac{4^n-1}{3}}\frac{\Delta^n}{1-\Delta\gamma^{4^n}}\eta \tag{9}$$

Proof. Let us now prove (8). Since $0 < a_0 < 1/2$, by lemma 2 $\{a_n\}$ is decreasing and the function

$$x\left(1+\frac{x}{2(1-x)}\right)$$

is increasing, therefore we have

$$a_n\left(1+\frac{a_n}{2(1-a_n)}\right) < a_0\left(1+\frac{a_0}{2(1-a_0)}\right) < 1.$$

In addition, as $y_n, x_n \in B(x_0, R\eta)$ for all $n \in \mathcal{N}$, then $y_n, x_n \in \Omega$, $n \in \mathcal{N}$. Thus (8) is true. Now we prove that $\{x_n\}$ is a Cauchy sequence. To do this, we consider $n, m \geq 1$:

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \ldots + \|x_{n+1} - x_n\| \\
&\leq \left(1 + \frac{a_n}{2(1-a_n)}\right)\eta \left[\prod_{j=0}^{n+m-2} f(a_j)g(a_j) + \ldots + \prod_{j=0}^{n-1} f(a_j)g(a_j)\right] \\
&\leq \left(1 + \frac{a_0}{2(1-a_0)}\gamma^{\frac{4^{n-1}-1}{3}}\right)\eta \left[\gamma^{\frac{4^{n+m-1}-1}{3}}\Delta^{n+m-1} + \ldots + \gamma^{\frac{4^n-1}{3}}\Delta^n\right] \\
&= \left(1 + \frac{a_0}{2(1-a_0)}\gamma^{\frac{4^n-1}{3}}\right)\eta\gamma^{\frac{4^n-1}{3}}\Delta^n \left[\gamma^{\frac{4^n(4^{m-1}-1)}{3}}\Delta^{m-1} + \ldots \right. \\
&\quad \left. + \gamma^{\frac{4^n(4-1)}{3}}\Delta + 1\right].
\end{aligned}$$

Using the well known inequality $(1+x)^k > 1 + kx$, we have

$$\|x_{n+m} - x_n\| \leq \left(1 + \frac{a_0}{2(1-a_0)}\gamma^{\frac{4^n-1}{3}}\right)\gamma^{\frac{4^n-1}{3}}\Delta^n \frac{1-\gamma^{4^n m}\Delta^m}{1-\gamma^{4^n}\Delta}\eta. \tag{10}$$

Then $\{x_n\}$ is a Cauchy sequence.Now by letting $m \to \infty$ in (10), we obtain (9). From (9) it is evident that x_n has R-order four at least. To prove that $F(x^*) = 0$, notice that $\|\Gamma_n F(x_n)\| \to 0$ as $n \to \infty$. As $\|F(x_n)\| \leq \|F'(x_n)\|\|\Gamma_n F(x_n)\|$ and $\{\|F'(x_n)\|\}$ is a bounded sequence, we deduce $\|F(x_n)\| \to 0$ and then $F(x^*) = 0$ by the continuity of F. To show the uniqueness, we suppose that there exists $y^* \in B(x_0, 2/M\beta - R\eta)\bigcap\Omega$ another solution of $F(x) = 0$. Then

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*).$$

Using the estimate

$$\begin{aligned}
&\|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\|dt \leq M\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\|dt \leq \\
&\qquad \leq M\beta \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)dt < \frac{M\beta}{2}(R\eta + \frac{2}{M\beta} - R\eta) = 1,
\end{aligned}$$

we can see that the operator $\int_0^1 F'(x^* + t(y^* - x^*))dt$ has an inverse and consequently, $y^* = x^*$. □

one evaluation of F'. The simplest cases are (13) with $a = \pm 1$. In this case the iteration leads to

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n), \\ x_{n+1} &= y_n - \frac{z_n - x_n}{F(z_n) - F(x_n)} F(y_n), \quad z_n = 2y_n - x_n. \end{aligned} \tag{14}$$

The iteration (14) requires evaluation of F at three points x_n, y_n and z_n and evaluation of $F'(x_n)$ at every iteration step. As comparison, we consider the iteration proposed by Ostrowski and Traub [4]

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n), \\ x_{n+1} &= y_n - \frac{y_n - x_n}{2F(y_n) - F(x_n)} F(y_n) \end{aligned} \tag{15}$$

which has R-order four convergence, but only need to calculate the derivative $F'(x_n)$. It is easy to verify that

$$\begin{aligned} \frac{z_n - x_n}{F(z_n) - F(x_n)} &= F'(y_n)^{-1} + O((y_n - x_n)^2) \\ \frac{y_n - x_n}{2F(y_n) - F(x_n)} &= F'(y_n)^{-1} + O((y_n - x_n)^2) \end{aligned}$$

under the Lipschitz continuity assumption of second derivative $F''(x)$. Thus the iterations (13), as well as (14), have the same order of convergence as (15).

4 Numerical experiments and comparison

In this section we apply our methods to solve some nonlinear equations and compare the performance of our methods with other methods, in order to check their effectiveness. For comparison we present the following methods [7]:

$$x_{n+1} = x_n + \frac{1}{f'(x_n)} \left[\frac{f(x_n)^2}{f(y_n) - f(x_n)} - \frac{f(y_n)^2}{f(x_n)} \right] \tag{16}$$

and [5]

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)}. \tag{17}$$

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and [5]

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First, we consider the following scalar equations:

Example 1: $sin^2x - x^2 + 1 = 0$

Example 2: $x^3 + 4x^2 - 10 = 0$

Example 3: $ln(x) = 0$

The different iteration methods are used for the examples 1-3 with stopping criterion $|f(x_n)| \leq \varepsilon = 10^{-15}$.

Table1. Comparison of the iteration number of various methods.

Examples	x_0	4th order methods				3rd order	2nd order
		(1)	(14)	(15)	(4) [7]	(4) [6]	NM
1	1	4	3	3	4	5	6
2	2	3	3	3	4	4	5
3	1.5	3	3	3	3	3	5

Next we consider following systems of equations.

Example 4:

$$F(x,y) = \begin{pmatrix} x^2 - y + 1 \\ x - cos(\frac{\pi}{2}y) \end{pmatrix} = 0$$

Example 5:

$$F(x,y) = \begin{pmatrix} x^3 - 3xy - 1 \\ 3x^2y - y^3 \end{pmatrix} = 0$$

Example 6:

$$F(x,y,z) = \begin{pmatrix} \frac{cosy - sinx}{z^x - \frac{1}{y}} \\ e^x - z^2 \end{pmatrix} = 0$$

Table2. Comparison of the iteration number of various methods.

Examples	x_0	(1)	(14)	(15)	(4) [6]
4	(-0.1,1.1)	3	4	4	3
5	(10,10)	6	6	7	8
6	(1,0.5,1.5)	3	6	7	4

In this work, we consider two step Newton-like method and its modifications. We proved Newton-Kantorovich type convergence theorem using recurrent relations to show that it has a R-order four convergence and obtained an error estimate. The proposed method and its modifications are compared to previously known higher order convergence methods to show that they have an equivalent performance.

References

[1] Ezquerro J.A., Hernandez M.A.,On Halley type iterations with free second derivative, J.Comput.Appl.Math 170 (2004) 455-459

[2] Hernandez M.A., *Second Derivative free variant of the Chebyshev method for nonlinear equations*, J.Optimization theory and applications 104(2000),501-515

[3] Hernandez M.A., Salanova M.A., *Modification of the Kantorovich assumptions for semilocal convergence of the Chebyshev*, J.Comput.Appl.Math. 126(2000) 131-143

[4] X.Mi, X.Wang, *An R-order four iteration in Banach space*, J.Computational analysis and applications, vol 1, No 3, 2005, 305-318

[5] T.Zhanlav, O.Chuluunbaatar, *High order convergent iteration methods for solving nonlinear equations,* Bulletin of Peoples Friendship University of Russia 4(2009), 47-55

[6] Babajee D.K.R., Dauhoo M.Z., *Analysis of two Chebyshev like methods free from second derivatives for solving nonlinear systems of equations,* J.Comput.Appl.Math. 233(2010) 2002-2012

[7] Kumar A.M., *A fourth order iterative method for solving nonlinear equations,* Applied Mathematics and Computation 211(2009) 383-391