Global Optimization Approach to Game Theory

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Abstract

We formulate the problem of finding a Nash equilibrium for non zero sum three-person games as a nonlinear programming problem. Global solutions are examined and numerical results are given.

1 Introduction

Game theory plays an important role in applied mathematics, economics and decision theory. There are many works devoted to game theory[2-7]. Most of them deals with zero sum two person games or nonzero sum two person games. Also, two person non zero sum game was studied in [5] by reducing it to D.C programming. This paper considers nonzero sum three person game. The paper is organized as follows. In Section 2, we formulate non zero sum three person game and show that it can be formulated as a global optimization problem with quadratic constraints. In Section 3, we provide with numerical experiments done for nonzero sum three person game. We formulate the problem of finding a Nash equilibrium for non zero sum three-person games as a nonlinear programming problem. Global solutions are examined and numerical results are given.

2 Nonzero Sum Three-person Game

Consider the three-person game in mixed strategies with matrices (A, B, C) for players 1,2 and 3.

$$A = (a_{ijk}),\ B = (b_{ijk}),\ C = (c_{ijk}),$$

$$i = 1, \dots, m,\ j = 1, \dots, n,\ k = 1, \dots, s$$

Denote by D_q the set

$$D_q = \{u \in R^q \,|\sum_{i=1}^{q} u_i = 1,\ u_i \geq 0,\ i = 1, \dots, q\}$$

A mixed strategy for player 1 is a vector $x = (x_1, x_2, \dots, x_m) \in D_m$ representing the probability that player 1 uses a strategy i. Similarly, the mixed strategies for players 2 and 3 are $y = (y_1, y_2, \dots, y_n) \in D_n$ and $z = (z_1, z_2, \dots, z_n) \in D_s$. Their expected payoffs are given by :

$$f_1(x, y, z) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}x_i y_j z_k$$

$$f_2(x, y, z) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}x_i y_j z_k$$

$$f_3(x, y, z) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}x_i y_j z_k$$

Definition 1. *A triple of mixed strategies $x^* \in D_m,\ y^* \in D_n,\ z^* \in D_s,$ is a Nash equilibrium if*

$$\begin{cases} f_1(x^*, y^*, z^*) \geq f_1(x, y^*, z^*),\ \forall x \in D_m \\ f_2(x^*, y^*, z^*) \geq f_2(x^*, y, z^*),\ \forall y \in D_n \\ f_3(x^*, y^*, z^*) \geq f_3(x^*, y^*, z),\ \forall z \in D_s. \end{cases}$$

It is clear that

$$f_1(x^*, y^*, z^*) = \max_{x \in D_m} f_1(x, y^*, z^*)$$

$$f_2(x^*, y^*, z^*) = \max_{y \in D_n} f_2(x^*, y, z^*)$$

$$f_3(x^*, y^*, z^*) = \max_{z \in D_s} f_3(x^*, y^*, z),$$

For further purpose, it is useful to formulate the following statement.

Theorem 1. *A triple strategy (x^*, y^*, z^*) is a Nash equilibrium if and only if*

$$\begin{cases} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^s a_{ijk} x_i^* y_j^* z_k^* \geq \sum_{j=1}^n \sum_{k=1}^s a_{ijk} y_j^* z_k^*, \ i = 1, 2, \ldots, m \\ \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^s b_{ijk} x_i^* y_j^* z_k^* \geq \sum_{j=1}^n \sum_{k=1}^s b_{ijk} x_i^* z_k^*, \ j = 1, 2, \ldots, n \\ \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^s c_{ijk} x_i^* y_j^* z_k^* \geq \sum_{j=1}^n \sum_{k=1}^s c_{ijk} x_i^* y_j^*, \ k = 1, 2, \ldots, s \end{cases} \tag{1}$$

Proof. Necessity: Assume that (x^*, y^*, z^*) is a Nash equilibrium. Then by Definition 1, we have

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^s a_{ijk} x_i^* y_j^* z_k^* \geq \sum_{j=1}^n \sum_{k=1}^s a_{ijk} x_i y_j^* z_k^*, \ \forall x \in D_m \tag{2}$$

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^s a_{ijk} x_i^* y_j^* z_k^* \geq \sum_{i=1}^m \sum_{k=1}^s a_{ijk} x_i^* y_j z_k^*, \ \forall y \in D_n \tag{3}$$

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^s a_{ijk} x_i^* y_j^* z_k^* \geq \sum_{i=1}^m \sum_{j=1}^n a_{ijk} x_i^* y_j^* z_k, \ \forall z \in D_s \tag{4}$$

In the first inequality (2), successively choose $x = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in each of the m spots, in (3) choose $y = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in each of the n spots, and in (4) choose $z = (0, 0, \ldots, 1, \ldots, 0)$ with 1 in each of the s spots. We can easily see that

$$f_1(x^*, y^*, z^*) \geq \sum_{j=1}^n \sum_{k=1}^s a_{ijk} y_j^* z_k^*, \ i = 1, \ldots, m$$

$$f_2(x^*, y^*, z^*) \geq \sum_{i=1}^m \sum_{k=1}^s b_{ijk} x_i^* z_k^*, \ j = 1, \ldots, n$$

$$f_3(x^*, y^*, z^*) \geq \sum_{i=1}^m \sum_{j=1}^n c_{ijk} x_i^* y_j^*, \ k = 1, \ldots, s$$

Sufficiency: Suppose that for a triple $(x^*, y^*, z^*) \in D_m \times D_n \times D_s$, conditions (1) are satisfied. We choose $x \in D_m,\ y \in D_n$ and $z \in D_s$ and multiply (1) by x, y and z respectively. We obtain

$$\sum_{e=1}^{m} x_e[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}x_i^* y_j^* z_k^*] \geq \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}x_i y_j^* z_k^*$$

$$\sum_{e=1}^{n} y_e[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}x_i^* y_j^* z_k^*] \geq \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}x_i^* y_j z_k^*$$

$$\sum_{e=1}^{m} z_e[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}x_i^* y_j^* z_k^*] \geq \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}x_i^* y_j^* z_k$$

Taking into account that $\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = \sum_{k=1}^{s} z_k = 1$ we have

$$f_1(x^*, y^*, z^*) \geq f_1(x, y^*, z^*),\ \forall x \in D_m$$

$$f_2(x^*, y^*, z^*) \geq f_2(x^*, y, z^*),\ \forall y \in D_n$$

$$f_3(x^*, y^*, z^*) \geq f_3(x^*, y^*, z),\ \forall z \in D_s$$

which shows that (x^*, y^*, z^*) is a Nash equilibrium. The proof is complete. □

Theorem 2. *A triple strategy (x^*, y^*, z^*) is a Nash equilibrium for the nonzero sum three-person game if and only if there exist scalars (p^*, q^*, t^*) such that $(x^*, y^*, z^*, p^*, q^*, t^*)$ is a solution to the following nonlinear programming problem :*

$$\max_{(x,y,z,p,q,t)} F(x, y, z, p, q, t) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}(a_{ijk} + b_{ijk} + c_{ijk})x_i y_j z_k - p - q - t \quad (5)$$

subject to :

$$\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}y_j z_k \leq p,\ i = 1, \ldots, m, \quad (6)$$

$$\sum_{i=1}^{m}\sum_{k=1}^{s} b_{ijk}x_i z_k \leq q,\ j = 1, \ldots, n, \quad (7)$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijk}x_i y_j \leq t,\ k = 1, \ldots, s, \quad (8)$$

$$\begin{aligned}
&\sum_{i=1}^{m} x_i = 1,\ x_i \geq 0,\ i = 1, \dots, m \\
&\sum_{j=1}^{n} y_j = 1,\ y_j \geq 0,\ j = 1, \dots, n \\
&\sum_{k=1}^{s} z_k = 1,\ z_k \geq 0,\ k = 1, \dots, s
\end{aligned} \tag{9}$$

Proof. Necessity: Now suppose that (x^*, y^*, z^*) is a Nash point. Choose scalars $p^*,\ q^*$ and t^* as: $p^* = f_1(x^*, y^*, z^*),\ q^* = f_2(x^*, y^*, z^*)$, and $t^* = f_3(x^*, y^*, z^*)$. We show that $(x^*, y^*, z^*, p^*, q^*, t^*)$ is a solution to problem (5)–(9).First, we show that $(x^*, y^*, z^*, p^*, q^*, t^*)$ is a feasible point for problem (5).

By Theorem 1, the equivalent characterization of a Nash point, we have

$$\begin{cases}
\sum_{j=1}^{n} \sum_{k=1}^{s} a_{ijk} y_j^* z_k^* \geq f_1(x^*, y^*, z^*) \\
\sum_{i=1}^{m} \sum_{k=1}^{s} b_{ijk} x_i^* z_k^* \geq f_2(x^*, y^*, z^*) \\
\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ijk} x_i^* y_j^* \geq f_3(x^*, y^*, z^*)
\end{cases}$$

The rest of the constraints are satisfied because $x \in D_m,\ y \in D_n$ and $z \in D_s$. It meant that $(x^*, y^*, z^*, p^*, q^*, t^*)$ is a feasible point. Choose any $x \in D_m,\ y \in D_n,\ z \in D_s$.

Multiply (6)–(8) by $x_i,\ y_j$ and z_k respectively. If we have sum up these inequalities, we obtain

$$f_1(x, y, z) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} a_{ijk} x_i y_j z_k \leq p$$

$$f_2(x, y, z) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} b_{ijk} x_i y_j z_k \leq q$$

$$f_3(x, y, z) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} c_{ijk} x_i y_j z_k \leq t$$

Hence, we get

$$F(x, y, z, p, q, t) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} (a_{ijk} + b_{ijk} + c_{ijk}) x_i y_j z_k - p - q - t \leq 0$$

for all $x \in D_m,\ y \in D_n$ and $z \in D_s$.

But with $p^* = f_1(x^*, y^*, z^*),\ q^* = f_2(x^*, y^*, z^*),$ and $t^* = f_3(x^*, y^*, z^*)$,

we have $F(x^*, y^*, z^*, p^*, q^*, t^*) = 0$ Hence, the point $(x^*, y^*, z^*, p^*, q^*, t^*)$ is a solution to the problem (5)–(9).

Sufficiency: Now we have to show reverse, namely, that any solution of problem (5)–(9) must be a Nash point. Let $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t})$ be any solution of problem (5)–(9). Let (x^*, y^*, z^*) be a Nash point for the game, and set $p^* = f_1(x^*, y^*, z^*),\ q^* = f_2(x^*, y^*, z^*),$ and $t^* = f_3(x^*, y^*, z^*)$.

We will show that $(\bar{x}, \bar{y}, \bar{z})$ must be a Nash equilibrium of the game. Since $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t})$ is a feasible point, we have

$$\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}\bar{y}_j\bar{z}_k \le \bar{p},\ i = 1, \ldots, m, \tag{10}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{s} b_{ijk}\bar{x}_i\bar{z}_k \le \bar{q},\ j = 1, \ldots, n, \tag{11}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ijk}\bar{x}_i\bar{y}_j \le \bar{t},\ k = 1, \ldots, s, \tag{12}$$

Hence, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} a_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k \le \bar{p}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} b_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k \le \bar{q}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s} c_{ijk}\bar{x}_i\bar{y}_j\bar{z}_k \le \bar{t}$$

Adding these inequalities, we obtain

$$F(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t}) = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}[a_{ijk} + b_{ijk} + c_{ijk}]\bar{x}_i\bar{y}_j\bar{z}_k - \bar{p} - \bar{q} - \bar{t} \le 0. \tag{13}$$

We know that at a Nash equilibrium $F(x^*, y^*, z^*, p^*, q^*, t^*) = 0$. Since $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t})$ is also a solution, $F(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t})$ be equal to zero:

$$F(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t}) = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} a_{ijk} \bar{x}_i \bar{y}_j \bar{z}_k - \bar{p} \right) + \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} b_{ijk} \bar{x}_i \bar{y}_j \bar{z}_k - \bar{q} \right) + \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} c_{ijk} \bar{x}_i \bar{y}_j \bar{z}_k - \bar{t} \right) = 0 \tag{14}$$

Consequently,

$$\begin{cases} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} a_{ijk} \bar{x}_i \bar{y}_j \bar{z}_k = \bar{p} \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} b_{ijk} \bar{x}_i \bar{y}_j \bar{z}_k = \bar{q} \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} c_{ijk} \bar{x}_i \bar{y}_j \bar{z}_k = \bar{t} \end{cases}$$

Since a point $(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{t})$ feasible, we can write the constrains (10)–(12) as follows:

$$\begin{cases} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} a_{ijk} \bar{x}_i \bar{y}_j \bar{z}_k \le \sum_{j=1}^{n} \sum_{k=1}^{s} a_{ijk} \bar{y}_j \bar{z}_k \quad i = 1, \dots, m \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} b_{ijk} \bar{x}_i \bar{y}_j \bar{z}_k \le \sum_{i=1}^{m} \sum_{k=1}^{s} b_{ijk} \bar{x}_i \bar{z}_k \quad j = 1, \dots, n \\ \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{s} c_{ijk} \bar{x}_i \bar{y}_j \bar{z}_k \le \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ijk} \bar{x}_i \bar{y}_j \quad k = 1, \dots, s \end{cases}$$

Now taking into account the above results, by Theorem 1 we conclude that $(\bar{x}, \bar{y}, \bar{z})$ is a Nash point which a completes the proof. □

3 Computational Experiments

Let $A = (a_{ijk})_{2\times2\times2}$ and $B = (b_{ijk})_{2\times2\times2}$, $C = (ijk)_{2\times2\times2}$

Three problems of type (5)–(9) have been solved numerically on "MATLAB" for dimensions $2 \times 2 \times 2$. In all cases, Nash points were found successfully. These problems were:

Problem 1. Let $a_{111} = 2, a_{112} = 3, a_{121} = -1, a_{122} = 0, a_{211} = 1, a_{212} = -2, a_{221} = 4, a_{222} = 3$, $b_{111} = 1, b_{112} = 2, b_{121} = 0, b_{122} = -1, b_{211} = -1, b_{212} = 0, b_{221} = 2, b_{222} = 1$, and $c_{111} = 3, c_{112} = 2, c_{121} = 1, c_{122} = -3, c_{211} = 0, c_{212} = 2, c_{221} =$

$-1, c_{222} = 2.$

Then we have the problem:

$$F(x,y,z,p,q,t) = 6x_1y_1z_1 + 7x_1y_1z_2 - 3x_1y_2z_2 + 5x_2y_1z_2 +$$

$$+6x_2y_2z_2 - p - q - t \to max$$

$$\begin{cases} 2y_1z_1 + 3y_1z_2 - y_2z_1 - p & \leq 0 \\ y_1z_1 - 2y_1z_2 + 4y_2z_1 + 3y_2z_2 - p & \leq 0 \\ x_1z_1 + 2x_1z_2 - x_2z_1 - q & \leq 0 \\ -1x_1z_2 + 2x_2z_1 + x_2z_2 - q & \leq 0 \\ 3x_1y_1 + x_1y_2 - x_2y_2 - t & \leq 0 \\ 2x_1y_1 - 3x_1y_2 + 2x_2y_1 + 2x_2y_2 - t & \leq 0 \\ x_1 + x_2 & = 1 \\ y_1 + y_2 & = 1 \\ z_1 + z_2 & = 1 \\ x_1 \geq 0 \,, x_2 \geq 0 \,, y_1 \geq 0 \,, y_2 \geq 0 & \\ z_1 \geq 0 \,, z_2 \geq 0 \,, p \geq 0 \,, q \geq 0 \; t \geq 0 & \end{cases}$$

Solution is $F^* = -2.2204e - 016$, $x^* = (0.5191; 0.4809)^T$, $y^* = (0.5888; 0.4112)^T$ and $z^* = (0.5382; 0.4618)^T$. $p^* = 1.2281$, $q^* = 0.5$ and $t^* = 0.9327$

Problem 2. Let $a_{111} = 5, a_{112} = 3, a_{121} = 6, a_{122} = 7, a_{211} = 0, a_{212} = 8, a_{221} = 2, a_{222} = 1$, $b_{111} = 2, b_{112} = 4, b_{121} = -1, b_{122} = 0, b_{211} = 3, b_{212} = 5, b_{221} = 4, b_{222} = 9$, and $c_{111} = 2, c_{112} = 0, c_{121} = -4, c_{122} = -1, c_{211} = -2, c_{212} = 6, c_{221} = 8, c_{222} = 9$.

Solution is $F^* = -0.00986$, $x^* = (0.8; 0.2)^T$, $y^* = (0.1; 0)^T$ and $z^* = (0.5; 0.5)^T$. $p^* = -2.2204e - 016$, $q^* = 3.2$ and $t^* = 1.2$

Problem 3. Let $a_{111} = 3, a_{112} = 2, a_{121} = 1, a_{122} = 5, a_{211} = 8, a_{212} = 4, a_{221} = 1, a_{222} = 3$, $b_{111} = 3, b_{112} = 2, b_{121} = 4, b_{122} = 0, b_{211} = 1, b_{212} = 8, b_{221} = 6, b_{222} = 6$, and $c_{111} = 3, c_{112} = 1, c_{121} = 9, c_{122} = 2, c_{211} = 4, c_{212} = 7, c_{221} = 2, c_{222} = 3$.

Solution is $F^* = 0,\ x^* = (1;0)^T,\ y^* = (1;0)^T$ and $z^* = (0;1)^T$. $p^* = 4,\ q^* = 8$ and $t^* = 7$

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