A Note on integral inequalities involving several r-convex functions

V.Adiyasuren

School of Mathematics and Computer Science,
National University of Mongolia, Mongolia

Abstract

The main aim of the present note is to establish new Hadamard
like integral inequalities involving several log-convex and r-convex functions.
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1 Introduction

The following inequality is well known in the literature as the Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

where $f : I \longrightarrow \mathbb{R}$ is a convex function on the interval I of real numbers $a, b \in I$ with $a < b$.

Theorem 2 ([2]). *Suppose f is a positive r-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \le F_r(f(a),f(b)).$$

If f is a positive r-concave function function, then the inequality is reversed.

The main purpose of this note is to stablish new inequalities like (1) involving several log-convex and r-convex functions.

2 Main Results

We start with the following theorem.

Theorem 3. *Let $f_j : I \longrightarrow (0,\infty),\ j=1,2,...,n$ be log-convex functions and $a,b\in I$ with $a<b$. If $f_j(a)\neq f_j(b),\ j=1,2,...,n,$ then*

$$\frac{n^2}{b-a}\int_a^b\left(\prod_{j=1}^n f_j(t)\right)dt \le \sum_{j=1}^n\left(\sum_{i=0}^{n-1}(f_j(a))^{n-1-i}(f_j(b))^i\right)\cdot L(f_j(a),f_j(b)) \tag{4}$$

where L is the logarithmic mean of positive real numbers.

Remark 1. *In the special case $n=1$, proved in [2].*

Proof. Let $n>1$. Since $f_j, j=1,2,...,n,$ are log-convex functions, for any $t\in[0;1]$ we have

$$f_j(ta+(1-t)b)\le (f_j(a))^t\cdot (f_j(b))^{1-t},\ j=1,2,...,n \tag{5}$$

It is easy to observe that

$$\int_a^b\left(\prod_{j=1}^n f_j(x)\right)dx = (b-a)\int_0^1\left(\prod_{j=1}^n f_j(ta+(1-t)b)\right)dt \tag{6}$$

Using the elementary inequality

$$a_1a_2\cdot ...\cdot a_n \le \frac{1}{n}\left(a_1^n+a_2^n+...+a_n^n\right) \qquad (a_1,a_2,...,a_n\ge 0),$$

Theorem 2 ([2]). *Suppose f is a positive r-convex function on $[a, b]$. Then*

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Proof. Let $n > 1$. Since $f_j, j = 1, 2, ..., n$, are log-convex functions, for any $t \in [0; 1]$ we have

$$f_j(ta + (1-t)b) \le (f_j(a))^t \cdot (f_j(b))^{1-t},\ j = 1, 2, ..., n \qquad (5)$$

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$$a_1 a_2 \cdot ... \cdot a_n \le \frac{1}{n}\left(a_1^n + a_2^n + ... + a_n^n\right) \qquad (a_1, a_2, ..., a_n \ge 0),$$

(5) on the right side of (6) and making the change of variable we have

$$\begin{aligned}
\int_a^b \left(\prod_{j=1}^n f_j(x)\right) dx \leq & \frac{b-a}{n} \int_0^1 \left(\sum_{j=1}^n (f_j(ta+(1-t)b))^n\right) dt \\
\leq & \frac{b-a}{n} \int_0^1 \left(\sum_{j=1}^n \left((f_j(a))^t \cdot (f_j(b))^{1-t}\right)^n\right) dt \\
= & \frac{b-a}{n} \sum_{j=1}^n (f_j(b))^n \int_0^1 \left(\frac{f_j(a)}{f_j(b)}\right)^{tn} dt \\
= & \frac{b-a}{n^2} \sum_{j=1}^n (f_j(b))^n \int_0^n \left(\frac{f_j(a)}{f_j(b)}\right)^{z} dz \\
= & \frac{b-a}{n^2} \sum_{j=1}^n \frac{(f_j(a))^n-(f_j(b))^n}{\ln f_j(a)-\ln f_j(b)} \\
= & \frac{b-a}{n^2} \sum_{j=1}^n \left(\sum_{i=0}^{n-1} (f_j(a))^{n-1-i}(f_j(b))^i\right) \cdot \frac{f_j(a)-f_j(b)}{\ln f_j(a)-\ln f_j(b)} \\
= & \frac{b-a}{n^2} \sum_{j=1}^n \left(\sum_{i=1}^{n-1} (f_j(a))^{n-1-i}(f_j(b))^i\right) \cdot L(f_j(a), f_j(b))
\end{aligned} \tag{7}$$

Rewriting (7) we get the required inequality in (4). The proof is complete. □

The following theorem also holds.

Theorem 4. *Let $f_j : I \longrightarrow (0,\infty), j = 1,2,...,n,$ be log-convex functions differentiable on the interval of real numbers $\overset{\circ}{I}$ (the interior of I) and $a, b \in \overset{\circ}{I}$ with $a < b$. Then the following inequality holds:*

$$\frac{n}{b-a} \int_a^b \left(\prod_{j=1}^n f_j(x)\right) dx$$

$$\geq \frac{1}{b-a}\sum_{j=1}^{n} f_j\left(\frac{a+b}{2}\right)\cdot\int_a^b \left(\prod_{\substack{j=1\\ i\neq j}}^{n} f_j(x)\exp\left(\frac{f_j'\left(\frac{a+b}{2}\right)}{f_j\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right)\right)dx \tag{8}$$

Proof. Since $f_j, j = 1, 2, ..., n$ are differentiable and log-convex function on $\overset{\circ}{I}$, we have that

$$(\ln\circ f_j)(x) - (\ln\circ f_j)(y) \geq (x-y)(\ln\circ f_j)'(y) \tag{9}$$

for all $x, y \in \overset{\circ}{I}$ gives that

$$\ln\left(\frac{f_j(x)}{f_j(y)}\right) \geq \frac{f_j'(y)}{f_j(y)}(x-y) \tag{10}$$

for all $x, y \in \overset{\circ}{I}$. That is

$$f_j(x) \geq f_j(y)\cdot\exp\left(\frac{f_j'(y)}{f_j(y)}(x-y)\right) \tag{11}$$

Multiplying both sides of (11) by $\prod\limits_{i=1, i\neq j}^{n} f_i(x)$ and adding the resulting inequalities we have

$$n\cdot\prod_{j=1}^{n} f_j(x) \geq \sum_{j=1}^{n}\left(\prod_{\substack{i=1\\ i\neq j}}^{n} f_i(x)f_j(y)\exp\left(\frac{f_j'(y)}{f_j(y)}(x-y)\right)\right) \tag{12}$$

Now, if we choose $y = \dfrac{a+b}{2}$, from (12) we obtain

$$n\cdot\prod_{j=1}^{n} f_j(x) \geq \sum_{j=1}^{n} f_j\left(\frac{a+b}{2}\right)\cdot\prod_{\substack{i=1\\ i\neq j}}^{n} f_i(x)\exp\left(\frac{f_j'\left(\frac{a+b}{2}\right)}{f_j\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right) \tag{13}$$

Integrating both sides of (13) with respect to x from a to b and dividing both sides of the resulting inequality by $b-a$ in, we get the desired (8). The proof is complete. $\square$

Theorem 5. *Suppose $f_j : I \longrightarrow (0,\infty), j = 1,2,...,n,$ are r-convex functions, $[a,b] \in I$ with $a < b$. If $f_j(a) \neq f_j(b), j = 1,2,...,n,$ then*

$$\frac{n}{b-a}\int_a^b \prod_{j=1}^n f_j(t)dt$$

$$\leq \begin{cases} \prod\limits_{j=1}^n \dfrac{r+1}{n+r} \cdot \dfrac{(f_j(b))^{n+r}-(f_j(a))^{n+r}}{(f_j(b))^{r+1}-(f_j(a))^{r+1}} \cdot F_r(f_j(a), f_j(b)), & r \neq -1, -n \\ \sum\limits_{j=1}^n F_{-1}\left((f_j(a))^n, (f_j(b))^n\right), & r = -n \\ \sum\limits_{j=1}^n \dfrac{1}{n-1} \cdot \dfrac{(f_j(b))^{n-1}-(f_j(a))^{n-1}}{(\ln\circ f_j)(b)-(\ln\circ f_j)(a)} \cdot F_{-1}(f_j(a), f_j(b)), & r=-1,\ n>1 \end{cases}$$

Proof. 1) Let $r \neq -1, -n$. The case $r = 0$ has been dealt with Theorem 3. Suppose that $r \neq 0, -1, -n$. By (3) we have

$$\begin{aligned} \int_a^b \left(\prod_{j=1}^n f_j(t)\right) dt =&(b-a)\int_0^1 \left(\prod_{j=1}^n f_j(sb+(1-s)a)\right) ds \\ \leq&\frac{b-a}{n}\int_0^1 \left(\sum_{j=1}^n (f_j(sb+(1-s)a))^n\right) ds \\ =&\frac{b-a}{n}\sum_{j=1}^n \int_0^1 (f_j(sb+(1-s)a))^n ds \\ \leq&\frac{b-a}{n}\sum_{j=1}^n \int_0^1 \left(s(f_j(b))^r+(1-s)(f_j(a))^r\right)^{n/r} ds \\ =&\frac{b-a}{n}\sum_{j=1}^n \int_{(f_j(a))^r}^{(f_j(b))^r} \frac{t^{n/r}}{(f_j(b))^r-(f_j(a))^r} dt \end{aligned}$$

If we solve the integral of the last expression we have

$$\int_{(f_j(a))^r}^{(f_j(b))^r} \frac{t^{n/r}}{(f_j(b))^r-(f_j(a))^r}dt = \frac{1}{(f_j(b))^r-(f_j(a))^r}\cdot\frac{r}{n+r}\cdot t^{(n+r)/r}\Big|_{(f_j(a))^r}^{(f_j(b))^r}$$
$$=\frac{r+1}{n+r}\cdot\frac{(f_j(b))^{n+r}-(f_j(a))^{n+r}}{(f_j(b))^{r+1}-(f_j(a))^{r+1}}\cdot\frac{r}{r+1}\cdot\frac{(f_j(b))^{r+1}-(f_j(a))^{r+1}}{(f_j(b))^r-(f_j(a))^r}.$$

Hence the original expression becomes

$$\int_a^b\left(\prod_{j=1}^n f_j(t)\right)dt \le \frac{b-a}{n}\sum_{j=1}^n\frac{r+1}{n+r}\cdot\frac{(f_j(b))^{n+r}-(f_j(a))^{n+r}}{(f_j(b))^{r+1}-(f_j(a))^{r+1}}\cdot F_r(f_j(b),f_j(a)).$$

2) Let $r=-n$.

$$\int_a^b\left(\prod_{j=1}^n f_j(t)\right)dt = (b-a)\int_0^1\left(\prod_{j=1}^n f_j(sb+(1-s)a)\right)ds$$
$$\le\frac{b-a}{n}\int_0^1\left(\sum_{j=1}^n(f_j(sb+(1-s)a))^n\right)ds$$
$$=\frac{b-a}{n}\sum_{j=1}^n\int_0^1(f_j(sb+(1-s)a))^n ds$$
$$\le\frac{b-a}{n}\sum_{j=1}^n\int_0^1\left(\frac{s}{(f_j(b))^n}+\frac{1-s}{(f_j(a))^n}\right)^{-1}ds$$
$$=\frac{b-a}{n}\sum_{j=1}^n\int_{1/(f_j(a))^n}^{1/(f_j(b))^n}\frac{1}{\frac{1}{(f_j(b))^n}-\frac{1}{(f_j(a))^n}}\cdot\frac{1}{t}dt$$
$$=\frac{b-a}{n}\sum_{j=1}^n(f_j(a))^n(f_j(b))^n\cdot\frac{1}{(f_j(a))^n-(f_j(b))^n}\cdot((\ln\circ(f_j)^n)(a)-(\ln\circ(f_j)^n)(b))$$
$$=\frac{b-a}{n}\sum_{j=1}^n F_{-1}(f_j^n(a),f_j^n(b))$$

3) let $r = -1, n > 1$. We have again

$$\int_a^b \left(\prod_{j=1}^n f_j(t) \right) dt \leq \frac{1}{n} \int_a^b \sum_{j=1}^n (f_j(t))^n dt$$
$$= \frac{b-a}{n} \sum_{j=1}^n \int_a^b (f_j(sb + (1-s)a))^n$$
$$\leq \frac{b-a}{n} \int_0^1 \sum_{j=1}^n \left(\frac{s}{f_j(b)} + \frac{(1-s)}{f_j(a)} \right)^{-n} ds$$
$$= \frac{b-a}{n} \sum_{j=1}^n \frac{1}{\frac{1}{f_j(b)} - \frac{1}{f_j(a)}} \cdot \int_{1/f_j(a)}^{1/f_j(b)} t^{-n} dt$$
$$= \frac{b-a}{n} \cdot \sum_{j=1}^n f_j(a) f_j(b) \cdot \frac{1}{f_j(a) - f_j(b)} \cdot \frac{1}{1-n} \cdot t^{1-n} \Big|_{1/f_j(a)}^{1/f_j(b)}$$
$$= \frac{b-a}{n} \sum_{j=1}^n \frac{1}{n-1} \cdot \frac{(f_j(b))^{n-1} - (f_j(a))^{n-1}}{(\ln \circ f_j)(b) - (\ln \circ f_j)(a)} \cdot f_j(a) f_j(b)$$
$$\cdot \frac{(\ln \circ f_j)(a) - (\ln \circ f_j)(b)}{f_j(a) - f_j(b)}$$
$$= \frac{b-a}{n} \sum_{j=1}^n \frac{1}{n-1} \cdot \frac{(f_j(b))^{n-1} - (f_j(a))^{n-1}}{(\ln \circ f_j)(b) - (\ln \circ f_j)(a)} \cdot F_{-1}(f_j(b), f_j(a))$$

The proof is complete. □

References

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