Convex-concave fractional minimization problem

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Abstract

We consider the convex-concave fractional minimization problem with an arbitrary feasible set. It has been shown that the problem can be treated as a quasiconvex minimization problem. For the case of convex feasible set, we reduce the original problem to pseudoconvex minimization problem showing that a local solution is global. We also show that a gradient method can be applied to this problem. Computational experiments have been done on some test problems.

1. Introduction

In this paper we consider the fractional programming problem:

$$\max_{x \in D} \frac{f(x)}{g(x)} \ , \tag{1.1}$$

where $D \subset R^n$ is a subset, and $f(x)$ is convex, $g(x)$ is concave on D, $f(x)$ and $g(x)$ positive on D. We call this problem as the convex-concave fractional minimization problem. Problem (1.1) has many applications in economics and engineering. For instance, problems such as minimization of average cost function [4] and minimizing the ratio between the amount of resource wasted and used on the production plan belong to a class of fractional programming.
The most well-known and studied class of fractional programming is the linear fractional programming class. When D is convex then well known existing methods for solving problem (1.1) are variable transformation [8], nonlinear programming approach [7], and parametric approach [5]. The variable transformation method reduces problem (1.1) to convex programming for the case

$$D = \{x \in S \subset R^n \mid h(x) \leq 0\}$$

with $h:\ R^n \to\ R^m$ a convex vector-valued function and S a convex set.

Theorem 1.1 [7] Problem (1.1) can be reduced to convex programming

$$\min\{tf(t^{-1}y) \mid th(t^{-1}y) \leq 0,\ tg(t^{-1}y),\ t^{-1}y \in S,\ t > 0\} \tag{1.2}$$

applying the transformation

$$y = xt \text{ and } t = \frac{1}{g(x)}\ .$$

Moveover, if (y^*, t^*) solves problem (1.2) then $x^* = t^{-1}y^*$ solves (1.1)
One of the most popular strategies for fractional programming is the parametric approach which considers the class of optimization problems associated with problem (1.1) given by

$$\inf_{x \in D}\{f(x) - \lambda g(x)\} \tag{1.3}$$

with $\lambda \in R$.
Introduce the function $F(\lambda)$ as follows

$$F(\lambda) = \min_{x \in D}\{f(x) - \lambda g(x)\}.$$

Lemma 1.1 [5] If D is a compact set then

(a) The function $F:\ R \to R$ is concave, continuous and strictly increasing.

(b) The optimal solution λ^* to (1.1) is finite and $F(\lambda^*) = 0$.

(c) $F(\lambda) = 0$ implies that $\lambda = \lambda^*$.

(d) $\lambda^* = \dfrac{f(x^*)}{g(x^*)} = \min_{x \in D} \dfrac{f(x)}{g(x)}$.

2. Global optimality conditions

Consider the convex-concave fractional minimization problem

$$\min_{x \in D} \varphi(x) \tag{2.1}$$

where, $\varphi(x) = \dfrac{f(x)}{g(x)}$, $D \subset R^n$ is an arbitrary compact set, $f, g:\ R^n \to D$ are differentiable functions, $f(x)$ is convex, $g(x)$ is concave on D, f and g are positive on D. Problem (2.1) belongs to a class of global optimization problems. Classical optimality conditions for problem (2.1) can not always guarantee finding global solutions.

In order to formulate global optimality conditions for (2.1), we need to introduce the set:

$$L(\varphi, C) = \{x \in D \mid \varphi(x) \leq C\}.$$

Clearly, $L(\varphi, C)$ is convex for all $C > 0$.

Definition 2.1 A function $h: \ D \to R$ is said to be quasiconvex on a convex set $D \subset R^n$ if

$$h(\alpha x + (1-\alpha)y) \leq \max\{h(x),\ h(y)\}$$

holds for all $x, y \in D$ and $\alpha \in [0, 1]$.

Lemma 2.1 [1] The function $h(x)$ is quasiconvex on D if and only if the set $L(h, C)$ is convex for all C.
Then it is obvious that $\varphi(x)$ is quasiconvex on D. Thus problem (2.1) reduces to quasiconvex minimization problem. Now we can apply the global optimality conditions [2] for problem (2.1)

Theorem 2.1 [1] Let z be a solution to problem (2.1), and let

$$E_C(\varphi) = \{y \in R^n \mid \varphi(y) = C\}.$$

Then

$$\langle \varphi'(x), x - y \rangle \geq 0 \ \text{ for all } y \in E_{\varphi(z)}(\varphi) \text{ and } x \in D. \tag{2.2}$$

If, in addition

$$\lim_{||x|| \to \infty} \varphi(x) = +\infty \text{ and } \varphi'(x + \alpha\varphi'(x)) \neq 0$$

holds for all $x \in D$ and $\alpha \geq 0$, then condition (2.2) becomes sufficient.
The optimality condition (2.2) can be written as follows:

$$\sum_{i=1}^{n} \left\{ \frac{\partial f(x)}{\partial x_i} g(x) - \frac{\partial g(x)}{\partial x_i} f(x) \right\} \left(\frac{x_i - y_i}{g^2(x)} \right) \geq 0$$

for all $y \in E_{\varphi(z)}(\varphi)$ and $x \in D$.

Lemma 2.2 If there exists a pair $(u, y) \in D \times E_{\varphi(z)}(\varphi)$ such that

$$\langle \varphi'(u), u - y \rangle < 0$$

then $\varphi(u) < \varphi(z)$.
Proof. On the contrary, assume that $\varphi(u) > \varphi(y) = \varphi(z)$. Since φ is quasiconvex, we have

$$\varphi(\alpha y + (1-\alpha)u) \leq \max\{\varphi(u), \varphi(y)\} = \varphi(u).$$

By Taylor's formula, there is a neighborhood of the point u which

$$\varphi(u + \alpha(y-u)) - \varphi(y) = \alpha\left[\langle \varphi'(u), y-u\rangle + \frac{o(\alpha||y-u||)}{\alpha}\right] \leq 0,$$

$\alpha > 0$. Taking into account that

$$\frac{o(\alpha||y-u||)}{\alpha} \xrightarrow{\alpha\to 0} 0,$$

we obtain $\langle\varphi'(u), y-u\rangle \leq 0$ or $\langle\varphi'(u), u-y\rangle \geq 0$ which contradicts $\langle\varphi'(u), u-y\rangle < 0$. This completes the proof.

Definition 2.1 Let Q be a subset of R^n. A differentiable function $h: Q \to R$ is pseudoconvex at $y \in Q$ if

$$h(x) - h(y) < 0 \text{ which implies } h'(y)(x-y) < 0, \ \forall x \in Q.$$

A function $h(.)$ is pseudoconvex on Q if it is pseudoconvex at each point $y \in Q$.

Lemma 2.3 Let D be a convex set in R^n. Let $f: D \to R$ be convex, differentiable and positive; Let $g: D \to R$ be concave, differentiable and positive. Then the function $\varphi(x) = \dfrac{f(x)}{g(x)}$ is pseudoconvex.
Proof. Take any point $y \in D$. Introduce the function $\psi: D \to R$ as follows:

$$\psi(x) = f(x)g(y) - g(x)f(y).$$

Since $g(y) > 0$ and $f(y) > 0$, $\psi(x)$ is convex and differentiable. Clearly, $\psi(y) = 0$. It is obvious that

$$\varphi(y) > \varphi(x) \text{ which is equivalent to } \psi(y) > \psi(x).$$

Since $\psi(.)$ is convex and differentiable, then we have

$$0 > \psi(x) - \psi(y) \geq \langle\psi'(y), x-y\rangle.$$

Taking into account that

$$\frac{\langle\psi'(y), x-y\rangle}{[g'(y)]^2} = \frac{\langle f'(y)g(y) - g'(y)f'(y), x-y\rangle}{[g'(y)]^2} = \langle\varphi'(y), x-y\rangle$$

we obtain implications

$$\varphi(y) > \varphi(x), \text{ hence we have } \langle \varphi'(y), x - y\rangle < 0$$

which prove the assertion.

Lemma 2.4 Let D be a convex set. Then any local minimizer x^* of $\varphi(x)$ on D is also a global minimizer.
Proof. On the contrary, assume that x^* is not a global minimizer. Then there exists a point $u \in D$ such that

$$\varphi(x^*) > \varphi(u). \tag{2.2}$$

Since D is a convex set,

$$x^* + \alpha(u - x^*) = \alpha u + (1-\alpha)x^* \in D, \ \forall \alpha : \ 0 < \alpha < 1.$$

By Taylor's expansion, we have

$$\varphi(x^* + \alpha(u - x^*)) = \varphi(x^*) + \alpha\langle \varphi'(x^*), u - x^*\rangle + o(\alpha||u - x^*||),$$

where $\lim\limits_{\alpha \to 0+} \dfrac{o(\alpha||u - x^*||}{\alpha} = 0$. Since x^* is a local minimizer of $\varphi(.)$ on D, there exists $0 < \alpha^* < 1$ so that

$$\varphi(x^* + \alpha(u - x^*)) - \varphi(x^*) > 0, \ \forall \alpha : \ 0 < \alpha < \alpha^*,$$

which implies

$$\langle \varphi'(x^*), u - x^*\rangle > 0.$$

Since $\varphi(.)$ is pseudoconvex, $\langle \varphi'(x^*), u - x^*\rangle > 0$ implies that $\varphi(u) > \varphi(x^*)$ contradicting (2.2) $\varphi(x^*) > \varphi(u)$. This completes the proof.
Lemma 2.4 allows us to apply gradient methods for solving problem (2.1).

3. Numerical methods and results

Consider problem (2.1) for the quadratic case:

$$\min_{x \in D} \left\{ \frac{f(x)}{g(x)} = \frac{\langle Ax, x\rangle + \langle b, x\rangle}{\langle Cx, x\rangle + \langle d, x\rangle + e} \right\}$$

where $D = \{x \in R^n \mid Bx \leq l\}$ is compact, A and C are matrices such that $A_{n\times n} > 0$, $C_{n\times n} < 0, f(x) > 0$ and $g(x) > 0$ on D.
The algorithm of conditional gradient method is the following.

Algorithm

Step1. Choose an arbitrary feasible point $x^0 \in D$ and *set* $k := 0$.
Step2. Solve the linear programming

$$\langle \varphi'(x^k), \bar{x}_k \rangle = \min_{x \in D} \langle \varphi'(x^k), \bar{x} \rangle.$$

Let $\bar{x}_k$ be a solution to the above problem.
Step3. Compute η_k :

$$\eta_k = \langle \varphi'(x^k), \bar{x}_k - x^k \rangle.$$

Step4. If $\eta_k = 0$ then x^k is a solution.
Step5. $x^{k+1} = x^k(\alpha_k),\ x^k(\alpha) = x^k + \alpha(\bar{x}_k - x^k),\ \alpha \in [0,1],$

$$f(x^k(\alpha_k)) = \min_{\alpha \in [0,1]} f(x^k(\alpha))$$

Step6. Set $k := k+1$ and go to Step2.

Theorem 3.1 [3] The sequence $\{x^k,\ k = 0,1,\dots\}$ generated by Algorithm is a minimizing sequence, i.e,

$$\lim_{k \to \infty} \varphi(x^k) = \min_{x \in D} \varphi(x).$$

The following problems have been solved numerically on MATLAB based on Algorithm.
Consider problem (2.1) with the following matrices.
Problem 1.

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix},\quad C = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix},\quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad d = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

Subject to constraint: $D = \{-1 \le x_1 \le 3,\ -2 \le x_2 \le 4\}$.
Solution: $x^* = (0.3712; 0.5282)$. Global value: $f(x^*) = 2.0092$.
Problem 2.

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 3 & 0 \\ 4 & 1 & 2 \end{pmatrix},\quad C = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ -1 & 1 & -1 \end{pmatrix},\quad b = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix},\quad d = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Subject to constraint: $D_1 = \{1 \le x_1 \le 3,\ 2 \le x_2 \le 5,\ 1 \le x_3 \le 4\}$.
Solution: $x^* = (1,2,1)$. Global value: $f(x^*) = 0.6747$.
The feasible set is $D_2 = \{-1 \le x_1 \le 3, -2 \le x_2 \le 4, -1 \le x_3 \le 5\}$.

Solution: $x^* = (-0.4709, -0.3723, -0.5008)$, Global value: $f(x^*) = 0.6662$.
Problem 3.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}, C = \begin{pmatrix} -2 & 1 & 1 & 2 \\ 1 & -1 & -1 & -2 \\ 1 & -1 & -3 & 1 \\ 2 & -2 & 1 & -9 \end{pmatrix}, b = \begin{pmatrix} 2 \\ -2 \\ 3 \\ -4 \end{pmatrix}, d = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -1 \end{pmatrix}.$$

The feasible set is $D = \{-2 \le x_1 \le 2, -1 \le x_2 \le 4, -1 \le x_3 \le 5, -3 \le x_4 \le 1\}$.
Solution: $x^* = (-0.2658, 0.1253, -0.1331, 0.1330)$.
Global value: $f(x^*) = 0.6665$.
Problem 4.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & -2 & -1 \\ 1 & -1 & -3 & 1 & 1 \\ 2 & -2 & 1 & -9 & -2 \\ 1 & -1 & 1 & -6 & -4 \end{pmatrix},$$

$$b = (1, -8, -3,\ 2,\ 5)', \quad d = (-2,\ 2,\ 5, -6,\ 4)'$$

The feasible set is $D = \{-3 \le x_1 \le 2, -2 \le x_2 \le 3, -1 \le x_3 \le 5, -1 \le x_4 \le 4, -4 \le x_5 \le 1\}$,
Solution: $x^* = (-0.9832, 2.9790, -0.2103, -0.9033, -0.3180)$.
Global value: $f(x^*) = 0, 6608$.
Problem 5.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 1 & 0 & 0 & 1 & -1 \\ 1 & -2 & 1 & 1 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 1 & 0 & 1 & 0 & -2 & 1 \\ -1 & 1 & 0 & 1 & 1 & -1 \end{pmatrix},$$

$$b = (1, -1,\ 2,\ 2,\ 3, -3)', \quad d = (3, -1,\ 1, -1,\ 2,\ 4)'$$

$$D = \left\{ \begin{array}{ll} -4 \le x_1 \le 1, & -2 \le x_2 \le 5 \\ -1 \le x_3 \le 3, & -3 \le x_4 \le 2 \\ -2 \le x_5 \le 4, & -5 \le x_6 \le 2 \end{array} \right\}$$

Solution: $x^* = (-4.0, 0.2709, 0.2555, 0.6716, -0.1537, 1.3453)$.
Global value: $f(x^*) = 0, 6641$.

Conclusion

We considered the convex-concave fractional minimization problem with an arbitrary feasible set. When the feasible set was nonconvex, we formulated the global optimality conditions. For the case of convex feasible set, we showed that the problem can be reduced to pseudoconvex minimization problem. We have also shown that classical gradient methods can be applied to our problem finding the global solution. Some test problems have been solved by the proposed algorithm.

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