Some New Inequalities of Hardy-Hilbert Type

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Abstract

In this paper, we establish some new inequalities of Hardy-Hilbert type integral inequality, whose kernel is the homogeneous function and the best constant factors are also derived.

1 Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ and $f(x), g(x) \geq 0$ satisfy

$$0 < \int_0^\infty f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(x)dx < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}, \tag{1}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < pq \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}, \tag{2}$$

where the constant factors $\pi/(\sin \pi/p)$ and pq are the best possible. Inequalities (1) and (2) are called Hardy-Hilbert's inequalities ([1]) and are important in analysis and their applications ([2]). In the recent years, many new inequalities similar to (1) have been established ([3]-[6]).

The main objective of this paper is to build some new inequalities of Hardy-Hilbert type, whose kernel is the homogeneous function with the best constant factors. As applications, some particular results are given.

2 Preliminary lemmas

In order to prove our assertions, we need the following lemmas.

Lemma 1. *Let p and q be conjugate parameters with $p > 1$, and let $\lambda, s, r > 0$ such that $s + r = \lambda$. If $k_\lambda(x, y) : \mathbb{R}^2_+ \to \mathbb{R}$ is non-negative homogeneous function of degree $-\lambda$, i.e. $k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x, y)$, then*

$$\omega_\lambda(s, x) = \varpi_\lambda(r, y) = \widetilde{C}_\lambda(s), \tag{3}$$

where

$$\omega_\lambda(s, x) = \int_0^\infty k_\lambda(x, y)y^{s-1}x^r dy,$$

$$\varpi_\lambda(r, y) = \int_0^\infty k_\lambda(x, y)x^{r-1}y^s dx,$$

and

$$\widetilde{C}_\lambda(s) = \int_0^\infty k_\lambda(1, u)u^{s-1}du.$$

Proof. Setting $u = y/x$, we find

$$\omega_\lambda(s, x) = \int_0^\infty k_\lambda(1, u)u^{s-1}du = \widetilde{C}_\lambda(s),$$

and for $y > 0$ letting $x = y/u$, it is easy to find that

$$\varpi_\lambda(r, y) = \int_0^\infty k_\lambda\left(\frac{y}{u}, y\right) y^s \frac{y^{r-1}}{u^{r-1}} \frac{y}{u^2} du = \int_0^\infty k_\lambda(1, u)u^{s-1}du = \widetilde{C}_\lambda(s),$$

equation (3) is valid. This completes proof of the lemma. □

Lemma 2 ([1])**.** *If $p > 1, \rho > 0$ and $f_\rho(x) = \frac{1}{\Gamma(\rho)}\int_0^x (x - t)^{\rho-1}f(t)dt$, then*

$$\int_0^\infty \left(\frac{f_\rho(x)}{x^\rho}\right)^p dx < \left\{\frac{\Gamma(1 - \frac{1}{p})}{\Gamma(\rho + 1 - \frac{1}{p})}\right\}^p \int_0^\infty f^p(x)dx, \tag{4}$$

unless $f \equiv 0$. The constant factor $\left\{\frac{\Gamma(1-\frac{1}{p})}{\Gamma(\rho+1-\frac{1}{p})}\right\}^p$ is the best possible.

3 Main Results

Theorem 3.1. *Assume $F(x) = \int_0^x (x - t)f(t)dt$ and $G(y) := \int_0^y (x - t)g(t)dt$. Let p and q be conjugate parameters with $p > 1$ and let $\lambda, s, r > 0$ such that $s + r = \lambda, k_\lambda(x, y)$ is non-negative homogeneous function of degree $-\lambda$ in $\mathbb{R}^2_+$.*

If

$$0 < \widetilde{C}_\lambda(s) < \infty, 0 < \int_0^\infty k_\lambda(1, u)u^{s-\frac{1}{\ell}-1}du < \infty, \ell = p, q$$

and $f(x), g(x) \geq 0$ *satisfy* $0 < \int_0^\infty f^p(x)dx < \infty, \quad 0 < \int_0^\infty g^q(x)dx < \infty$, *then the following two inequalities hold:*

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}-2} F(x)G(y)dxdy$$
$$< C_\lambda(s,p,q) \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}} \quad (5)$$

and

$$\int_0^\infty \left(\int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}} F(x)dx \right)^p dy < \widetilde{C}_\lambda^p(s) \left(\frac{p^2}{p+1} \right)^p \int_0^\infty f^p(x)dx, \quad (6)$$

where the constant factors $C_\lambda(s,p,q) = \widetilde{C}_\lambda(s) \frac{(pq)^2}{(1+p)(1+q)}$ *and* $\widetilde{C}_\lambda^p(s) \left(\frac{p^2}{p+1} \right)^p$ *are the best possible.*

Proof. We set $J = \int_0^\infty \int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}-2} F(x)G(y)dxdy$. By Hölder's inequality and Lemma 1, we have

$$\begin{aligned} J &= \int_0^\infty \int_0^\infty k_\lambda(x,y)(y^{\frac{s-1}{p}} x^{\frac{r}{p}-2} F(x))(x^{\frac{r-1}{q}} y^{\frac{s}{q}-2} G(y))dxdy \\ &\leq \left\{ \int_0^\infty \int_0^\infty k_\lambda(x,y) y^{s-1} x^r \left(\frac{F(x)}{x^2} \right)^p dxdy \right\}^{\frac{1}{p}} \\ &\quad \cdot \left\{ \int_0^\infty \int_0^\infty k_\lambda(x,y) x^{r-1} y^s \left(\frac{G(y)}{y^2} \right)^q dxdy \right\}^{\frac{1}{q}} \\ &= \widetilde{C}_\lambda(s) \left\{ \int_0^\infty \left(\frac{F(x)}{x^2} \right)^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \left(\frac{G(y)}{y^2} \right)^q dy \right\}^{\frac{1}{q}}. \end{aligned}$$

Then by inequality (4) with $\rho = 2$, (5) is valid.

Supposing there exists a positive constant $C < C_\lambda(s,p,q)$, such that (5) is still valid when $C_\lambda(s,p,q)$ is replaced by C and for $n > \max\left\{ \frac{1}{p-1}, \frac{1}{q-1} \right\}, n \in \mathbb{N}$, setting $\widetilde{f}(x), \widetilde{g}(y)$ as follows:

$$\widetilde{f}(x) = \begin{cases} 0, & \text{for } x \in (0,1) \\ x^{-\frac{1+(1/n)}{p}}, & \text{for } x \in [1,\infty) \end{cases},$$

$$\widetilde{g}(y) = \begin{cases} 0, & \text{for } y \in (0,1) \\ y^{-\frac{1+(1/n)}{q}}, & \text{for } y \in [1,\infty) \end{cases},$$

then

$$C \left\{ \int_0^\infty \widetilde{f}^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \widetilde{g}^q(x)dx \right\}^{\frac{1}{q}} = nC, \quad (7)$$

and

$$\widetilde{F}(x) = \begin{cases} 0, & \text{for } x \in (0,1) \\ \varphi(n,p)\psi(n,p)x^{2-\frac{1+(1/n)}{p}} - \varphi(n,p)x + \psi(n,p), & \text{for } x \in [1,\infty) \end{cases},$$

$$\widetilde{G}(y)=\begin{cases}0, & \text{for } y\in(0,1)\\ \varphi(n,q)\psi(n,q)y^{2-\frac{1+(1/n)}{q}}-\varphi(n,q)y+\psi(n,q), & \text{for } y\in[1,\infty)\end{cases},$$

where $\varphi(n,\ell)=\frac{1}{1-\frac{1+(1/n)}{\ell}}$ and $\psi(n,\ell)=\frac{1}{2-\frac{1+(1/n)}{\ell}}$, $\ell=p,q$. Then

$$[\varphi(n,p)-\psi(n,p)][\varphi(n,q)-\psi(n,q)]\to\frac{(pq)^2}{(1+p)(1+q)},$$

as $n\to\infty$ and for $x,y\geq 1$,

$$\begin{aligned}\widetilde{F}(x)\widetilde{G}(y) &> \left([\varphi(n,p)-\psi(n,p)]x^{2-\frac{1+(1/n)}{p}}-\varphi(n,p)x\right)\\ &\quad\cdot\left([\varphi(n,q)-\psi(n,q)]y^{2-\frac{1+(1/n)}{q}}-\varphi(n,q)y\right)\\ &> [\varphi(n,p)-\psi(n,p)][\varphi(n,q)-\psi(n,q)]x^{2-\frac{1+(1/n)}{p}}y^{2-\frac{1+(1/n)}{q}}\\ &\quad-\varphi(n,q)[\varphi(n,p)-\psi(n,p)]x^{2-\frac{1+(1/n)}{p}}y-\varphi(n,p)[\varphi(n,q)-\psi(n,q)]xy^{2-\frac{1+(1/n)}{q}}.\end{aligned}$$

Then

$$\begin{aligned}J(n) &= \int_0^\infty\int_0^\infty k_\lambda(x,y)x^{r-\frac{1}{q}-2}y^{s-\frac{1}{p}-2}\widetilde{F}(x)\widetilde{G}(y)dxdy\\ &> [\varphi(n,p)-\psi(n,p)][\varphi(n,q)-\psi(n,q)]\int_1^\infty\int_1^\infty k_\lambda(x,y)x^{r-\frac{1}{np}-1}y^{s-\frac{1}{nq}-1}dxdy\\ &\quad-\varphi(n,q)[\varphi(n,p)-\psi(n,p)]\int_1^\infty\int_1^\infty k_\lambda(x,y)x^{r-\frac{1}{np}-1}y^{s-\frac{1}{p}-1}dxdy\\ &\quad-\varphi(n,p)[\varphi(n,q)-\psi(n,q)]\int_1^\infty\int_1^\infty k_\lambda(x,y)x^{r-\frac{1}{q}-1}y^{s-\frac{1}{nq}-1}dxdy\\ &= [\varphi(n,p)-\psi(n,p)][\varphi(n,q)-\psi(n,q)]I_1-\varphi(n,q)[\varphi(n,p)-\psi(n,p)]I_2\\ &\quad-\varphi(n,p)[\varphi(n,q)-\psi(n,q)]I_3.\end{aligned}$$

We set $I_1=\int_1^\infty\int_1^\infty k_\lambda(x,y)x^{r-\frac{1}{np}-1}y^{s-\frac{1}{nq}-1}dxdy$. Taking $u=y/x$ and by Fubini's theorem, we obtain

$$\begin{aligned}I_1 &= \int_1^\infty x^{-1-\frac{1}{n}}\left(\int_1^\infty k_\lambda(x,y)y^{s-\frac{1}{nq}-1}x^{r+\frac{1}{nq}}dy\right)dx\\ &= \int_1^\infty x^{-1-\frac{1}{n}}\left(\int_{1/x}^1 k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du+\int_1^\infty k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du\right)dx\\ &= n\int_1^\infty k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du+\int_1^\infty x^{-1-\frac{1}{n}}dx\int_{1/x}^1 k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du\\ &= n\int_1^\infty k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du+\int_0^1 k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du\int_{1/u}^\infty x^{-1-\frac{1}{n}}dx\\ &= n\left(\int_1^\infty k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du+\int_0^1 k_\lambda(1,u)u^{s+\frac{1}{np}-1}du\right).\end{aligned}$$

We denote by $I_2 = \int_1^\infty \int_1^\infty k_\lambda(x,y)x^{r-\frac{1}{np}-1}y^{s-\frac{1}{p}-1}dxdy$. Again taking $u = y/x$, we obtain

$$\begin{aligned} I_2 &= \int_1^\infty \int_0^\infty k_\lambda(x,y)x^{r-\frac{1}{np}-1}y^{s-\frac{1}{p}-1}dxdy - \int_1^\infty \int_0^1 k_\lambda(x,y)x^{r-\frac{1}{np}-1}y^{s-\frac{1}{p}-1}dxdy \\ &< \int_1^\infty x^{-1-(\frac{1}{p}+\frac{1}{np})}dx \int_0^\infty k_\lambda(1,u)u^{s-\frac{1}{p}-1}du \\ &= \frac{pn}{n+1}\int_0^\infty k_\lambda(1,u)u^{s-\frac{1}{p}-1}du < \infty. \end{aligned}$$

Similarly, we get

$$\begin{aligned} I_3 &:= \int_1^\infty \int_1^\infty k_\lambda(x,y)x^{r-\frac{1}{q}-1}y^{s-\frac{1}{nq}-1}dxdy \\ &< \frac{qn}{n+1}\int_0^\infty k_\lambda(1,u)u^{r-\frac{1}{q}-1}du < \infty. \end{aligned}$$

Hence by (7), we have

$$\begin{aligned} &\int_1^\infty [\varphi(n,p) - \psi(n,p)][\varphi(n,q) - \psi(n,q)]k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du \\ &\quad + \int_0^1 [\varphi(n,p) - \psi(n,p)][\varphi(n,q) - \psi(n,q)]k_\lambda(1,u)u^{s+\frac{1}{np}-1}du \\ &\quad - \frac{\varphi(n,q)[\varphi(n,p) - \psi(n,p)] + \varphi(n,p)[\varphi(n,q) - \psi(n,q)]}{n} \bigcirc(1) < C. \end{aligned}$$

Then by Fatou lemma, we have

$$\begin{aligned} C_\lambda(s,p,q) &= \frac{(pq)^2}{(1+p)(1+q)}\int_0^\infty k_\lambda(1,u)u^{s-1}du \\ &= \int_1^\infty \lim_{n\to\infty}[\varphi(n,p) - \psi(n,p)][\varphi(n,q) - \psi(n,q)]k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du \\ &\quad + \int_0^1 \lim_{n\to\infty}[\varphi(n,p) - \psi(n,p)][\varphi(n,q) - \psi(n,q)]k_\lambda(1,u)u^{s+\frac{1}{np}-1}du \\ &\quad - \lim_{n\to\infty}\frac{\varphi(n,q)[\varphi(n,p) - \psi(n,p)] + \varphi(n,p)[\varphi(n,q) - \psi(n,q)]}{n} \bigcirc(1) \\ &\le \varliminf_{n\to\infty}\Big(\int_1^\infty [\varphi(n,p) - \psi(n,p)][\varphi(n,q) - \psi(n,q)]k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du \\ &\quad + \int_0^1 [\varphi(n,p) - \psi(n,p)][\varphi(n,q) - \psi(n,q)]k_\lambda(1,u)u^{s+\frac{1}{np}-1}du \\ &\quad - \frac{\varphi(n,q)[\varphi(n,p) - \psi(n,p)] + \varphi(n,p)[\varphi(n,q) - \psi(n,q)]}{n} \bigcirc(1)\Big) \le C. \end{aligned}$$

Hence, the constant factor $C = C_\lambda(s,p,q)$ is the best possible.

We set $L(y) = \int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}} F(x)dx$. By Hölder's inequality and Lemma 1, we get

$$\begin{aligned} L(y) &= \int_0^\infty k_\lambda(x,y)(x^{\frac{r}{p}-2} y^{\frac{s-1}{p}} F(x))(x^{\frac{r-1}{q}} y^{\frac{s}{q}})dx \\ &\leq \left\{\int_0^\infty k_\lambda(x,y) x^{r-2p} y^{s-1} F^p(x)dx\right\}^{1/p} \left\{\int_0^\infty k_\lambda(x,y) x^{r-1} y^s dx\right\}^{1/q} \\ &= (\widetilde{C}_\lambda(s))^{\frac{1}{q}} \left\{\int_0^\infty k_\lambda(x,y) x^{r-2p} y^{s-1} F^p(x)dx\right\}^{1/p}. \end{aligned}$$

Hence again applying Lemma 1, we have

$$\begin{aligned} \int_0^\infty L^p(y)dy &\leq (\widetilde{C}_\lambda(s))^{\frac{p}{q}} \int_0^\infty \left(\int_0^\infty k_\lambda(x,y) x^r y^{s-1} dy\right) x^{-2p} F^p(x)dx \\ &= (\widetilde{C}_\lambda(s))^p \int_0^\infty \left(\frac{F(x)}{x^2}\right)^p dx. \end{aligned}$$

Then by inequality (4) with $\rho = 2$, (6) is valid.

If the constant factor $\widetilde{C}_\lambda^p(s)\left(\frac{p^2}{p+1}\right)^p$ in (6) is not the best possible, then there exists a positive constant K such that $K < \widetilde{C}_\lambda(s)\left(\frac{p^2}{p+1}\right)$ and (6) still remains valid if $\widetilde{C}_\lambda^p(s)\left(\frac{p^2}{p+1}\right)^p$ is replaced by K^p. Then by Hölder's inequality, (6) and by (4), we obtain

$$\begin{aligned} J &= \int_0^\infty \left(\int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}} F(x)dx\right)\left(\frac{G(y)}{y^2}\right)dy \\ &\leq \left\{\int_0^\infty \left(\int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}} F(x)dx\right)^p dy\right\}^{1/p} \left\{\int_0^\infty \left(\frac{G(y)}{y^2}\right)^q dy\right\}^{1/q} \\ &< \frac{q^2}{q+1} K \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \end{aligned}$$

which results that the constant factor $C_\lambda(s,p,q)$ in (5) is not the best possible. This contradiction shows that the constant factor $\widetilde{C}_\lambda^p(s)\left(\frac{p^2}{p+1}\right)^p$ in (6) is the best possible. The theorem is proved. □

If $k_\lambda(x,y) = 1/(x+y)^\lambda, 1/\max\{x^\lambda, y^\lambda\}$ or $1/|x-y|^\lambda$ then we obtain the following corollaries correspondingly. Let us denote

$$F(x) = \int_0^x (x-t)f(t)dt \quad \text{and} \quad G(y) = \int_0^y (x-t)g(t)dt.$$

Corollary 3.2. *Let p and q be conjugate parameters with $p > 1$ and let $\lambda, s, r > 0$, such that $r + s = \lambda, f(x), g(x) \geq 0$. If $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then the* *following two inequalities hold:*

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}-2}}{(x+y)^\lambda} F(x)G(y)dxdy$$
$$< B(r,s)\frac{(pq)^2}{(1+p)(1+q)} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}$$

and

$$\int_0^\infty \left(\int_0^\infty \frac{x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}}}{(x+y)^\lambda} F(x)dx\right)^p dy < [B(r,s)]^p \left(\frac{p}{p+1}\right)^p \int_0^\infty f^p(x)dx,$$

where the constant factors $B(r,s)\frac{(pq)^2}{(1+p)(1+q)}$ *and* $[B(r,s)]^p\left(\frac{p}{p+1}\right)^p$ *are the best possible.*

We set

$$F(x) = \int_0^x (x-t)f(t)dt \quad \text{and} \quad G(y) = \int_0^y (x-t)g(t)dt.$$

Corollary 3.3. *Let* p *and* q *be conjugate parameters with* $p > 1$ *and let* $\lambda, s, r > 0$*, such that* $r+s = \lambda, f(x), g(x) \geq 0$*. If* $0 < \int_0^\infty f^p(x)dx < \infty$ *and* $0 < \int_0^\infty g^q(x)dx < \infty$*, then the following two inequalities hold:*

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}-2}}{\max\{x^\lambda, y^\lambda\}} F(x)G(y)dxdy$$
$$< \frac{\lambda}{rs}\frac{(pq)^2}{(1+p)(1+q)} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}$$

and

$$\int_0^\infty \left(\int_0^\infty \frac{x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}}}{\max\{x^\lambda, y^\lambda\}} F(x)dx\right)^p dy < \left(\frac{\lambda}{rs}\right)^p \left(\frac{p^2}{p+1}\right)^p \int_0^\infty f^p(x)dx,$$

where the constant factors $\frac{\lambda}{rs}\frac{(pq)^2}{(1+p)(1+q)}$ *and* $\left(\frac{\lambda}{rs}\right)^p\left(\frac{p^2}{p+1}\right)^p$ *are the best possible.*

We set

$$F(x) = \int_0^x (x-t)f(t)dt \quad \text{and} \quad G(y) = \int_0^y (x-t)g(t)dt.$$

Corollary 3.4. *Let* p *and* q *be conjugate parameters with* $p > 1$ *and let* $0 < \lambda < 1, s, r > 0$*, such that* $r+s = \lambda, f(x), g(x) \geq 0$*. If* $0 < \int_0^\infty f^p(x)dx < \infty$ *and* $0 < \int_0^\infty g^q(x)dx < \infty$*, then the following two inequalities hold:*

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}-2}}{|x-y|^\lambda} F(x)G(y)dxdy$$
$$< (B(s,1-\lambda)+B(r,1-\lambda))\frac{(pq)^2}{(1+p)(1+q)} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}$$

and

$$\int_0^\infty \left(\int_0^\infty \frac{x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}}}{|x-y|^\lambda} F(x) dx \right)^p dy$$
$$< (B(s,1-\lambda) + B(r,1-\lambda))^p \left(\frac{p^2}{p+1} \right)^p \int_0^\infty f^p(x) dx,$$

where the constant factors $((B(s,1-\lambda)+B(r,1-\lambda))\frac{(pq)^2}{(1+p)(1+q)}$ *and* $(B(s,1-\lambda)+B(r,1-\lambda))^p\left(\frac{p^2}{p+1}\right)^p$ *are the best possible.*

Theorem 3.5. *Let* p *and* q *be conjugate parameters with* $p > 1$ *and let* $\lambda, s, r > 0$ *such that* $s + r = \lambda, k_\lambda(x, y)$ *is non-negative homogeneous function of degree* $-\lambda$ *in* $\mathbb{R}^2_+$. *Let* $\{\alpha_k\}_{k=1}^\infty, \{\beta_k\}_{k=1}^\infty$ *be positive sequences such* $\sum_{k=1}^\infty \alpha_k = 1, \sum_{k=1}^\infty \beta_k = 1$ *and* $\{f_k\}_{k=1}^\infty, \{g_k\}_{k=1}^\infty$ *be sequences of nonnegative integrable functions and let*

$$F_k(x) = \int_0^x (x-t) f_k(t) dt \quad \textit{and} \quad G_k(y) = \int_0^y (x-t) g_k(t) dt, \quad k = 1, 2, \ldots.$$

If

$$0 < \widetilde{C}_\lambda(s) < \infty, 0 < \int_0^\infty k_\lambda(1,u) u^{s-\frac{1}{l}-1} du < \infty, l = p, q$$

and

$$0 < \int_0^\infty \left(\sum_{k=1}^\infty \alpha_k f_k(x) \right)^p dx < \infty, \quad 0 < \int_0^\infty \left(\sum_{k=1}^\infty \beta_k g_k(x) \right)^q dx < \infty,$$

then the following two inequalities hold:

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}-2} \left(\prod_{k=1}^\infty F_k^{\alpha_k}(x) \right) \left(\prod_{k=1}^\infty G_k^{\beta_k}(y) \right) dx dy$$
$$< C_\lambda(s,p,q) \left\{ \int_0^\infty \left(\sum_{k=1}^\infty \alpha_k f_k(x) \right)^p dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \left(\sum_{k=1}^\infty \beta_k g_k(x) \right)^q dx \right\}^{\frac{1}{q}} \tag{8}$$

and

$$\int_0^\infty \left(\int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-2} y^{s-\frac{1}{p}} \left(\prod_{k=1}^\infty F_k^{\alpha_k}(x) \right) dx \right)^p dy$$
$$< \widetilde{C}^p_\lambda(s) \left(\frac{p}{p+1} \right)^p \int_0^\infty \left(\sum_{k=1}^\infty \alpha_k f_k(x) \right)^p dx, \tag{9}$$

where the constant factors $C_\lambda(s,p,q) = \widetilde{C}_\lambda(s)\frac{(pq)^2}{(1+p)(1+q)}$ *and* $\widetilde{C}^p_\lambda(s)\left(\frac{p^2}{p+1}\right)^p$ *are the best possible.*

Proof. According to the Arithmetic-Geometric Mean inequality

$$\prod_{k=1}^{\infty} h_k^{\alpha_k}(x) \leq \sum_{k=1}^{\infty} \alpha_k h_k(x),$$

we have that

$$\prod_{k=1}^{\infty} F_k^{\alpha_k}(x) \leq \sum_{k=1}^{\infty} \alpha_k F_k(x) = \int_0^x (x-t) \left(\sum_{k=1}^{\infty} \alpha_k f_k(t) \right) dt, \tag{10}$$

$$\prod_{k=1}^{\infty} G_k^{\beta_k}(x) \leq \sum_{k=1}^{\infty} \beta_k G_k(x) = \int_0^x (x-t) \left(\sum_{k=1}^{\infty} \beta_k g_k(t) \right) dt. \tag{11}$$

By using inequality (4) with the functions $\sum_{k=1}^{\infty} \alpha_k f_k(t)$, $\sum_{k=1}^{\infty} \beta_k g_k(t), \rho = 2$ and (10), (11) the inequality (8) is proved. The constant factor in the inequality (8) is the best possible since by applying it with $f_k(t) = f(t)$, $g_k(t) = g(t)$, $k = 1, 2, \ldots$, it reduces to (5) and it is known the constant factor in inequality (5) is best possible.

The proof of inequality (9) is similar to the proof of inequality (8). The theorem is proved. □

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