The local growth envelope of Sobolev-Orlicz space

J. Davaadulam

School of Mathematics and Computer Science, National University of Mongolia, Ulaanbaatar, Mongolia

Abstract

This paper aim is to give sharp estimate for the local growth envelope function of the Sobolev-Orlicz space.

Keywords: Embedding, local growth envelope function, Sobolev-Orlicz space

1 Introduction

The researches of Sobolev embeddings begin with the classical Sobolev theorems. One knows from the Sobolev embedding theorem that given $k \in \mathbb{N}$ and $1 < p < \infty$,

$$W_p^k \hookrightarrow L_\infty \quad \text{if and only if,} \quad k > \frac{n}{p},$$

a result which has been generalized to the generalized Sobolev spaces, where the role of the "basic space" plays more general than L_p, rearrangement invariant space $E = E(\Omega)$; here $\Omega \subset \mathbb{R}^n$ is bounded domain.

In this paper we consider so called Sobolev-Orlicz space, where the rearrangement invariant space is weighted Orlicz-Lorentz space.

In the third section we present a description of associated space for the cone of all nonnegative decreasing functions from Orlicz space $L_{\Phi,\nu}(0, S)$, where $S < \infty$. From this description follow the main results in this paper. Also it might have some interest independently of the main topic under consideration in this paper.

In the fourth section we give the necessary and sufficient condition of embedding of Sobolev-Orlicz space in L_∞. Note that the embedding in L_∞ imply the essential boundedness of the functions involved. In such spaces which are not embedded in L_∞ the unboundedness is measured by the determination of so called local growth envelope function. So in the fifth section we give the sharp estimate for the local growth envelope function of the Sobolev-Orlicz space.

2 Main notations and definitions.

As usual, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space and $\mathbb{R}_+ = (0, +\infty)$. We use the equivalence $\sim$ in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1 a_k \le b_k \le c_2 a_k \quad \text{or} \quad c_1\varphi(x) \le \psi(x) \le c_2\varphi(x)$$

for all admitted values of the discrete variables k or the continuous variable x, where $\{a_k\}$, $\{b_k\}$ are non-negative sequences and φ, ψ are non-negative functions.

Recall that Banach function space is a Banach space of measurable functions in which the norm is related to the underlying measure in an appropriate way. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function and f^* its decreasing rearrangement, that is decreasing right continuous function on $\mathbb{R}_+$, equi-measurable with $|f|$:

$$\mu_n\{x \in \mathbb{R}^n : |f(x)| > y\} = \mu_1\{t \in \mathbb{R}_+ : f^*(t) > y\}; \quad y > 0,$$

where μ_n and μ_1 are the Lebesgue measures on $\mathbb{R}^n$ and on $\mathbb{R}_+$ respectively.

Banach function space is called rearrangement invariant space (RIS) if its norm is monotone with respect to rearrangements.

It is known the Luxembourg representation for the norm of function in RIS trough the norm of its rearrangement([1]): there exists unique RIS $\widetilde{E}(R_+)$ for RIS $E(\mathbb{R}^n)$ such that

$$\|f\|_{E(\mathbb{R}^n)} = \|f^*\|_{\widetilde{E}(R_+)}.$$

Given two Banach spaces X and Y, we write

$$X \hookrightarrow Y$$

if $X \subset Y$ and the natural embedding of X in Y is continuous.

3 Definition of the Sobolev-Orlicz space.

Let $\phi : [0, \infty) \to [0, \infty)$ be increasing and left continuous function, with $\phi(0) = 0$. Suppose on $(0, \infty)$ that ϕ is neither identically zero nor identically infinite. Then the function Φ defined by

$$\Phi(t) = \int_0^t \phi(s)ds, \quad (0 \le t \le \infty) \tag{3.1}$$

is said to be Young's function. Let also

$$\psi(s) = \inf\{u : \phi(u) \ge s\}, \quad (0 \le s \le \infty).$$

Then the function

$$\Psi(t) = \int_0^t \psi(s)ds, \quad (0 \le t \le \infty) \tag{3.2}$$

is called the complementary Young's function of Φ.

Let ν be non negative measurable function on $(0, \infty)$ such that

$$0 < V(t) = \int_0^t \nu(s) ds < \infty, \quad (0 < t \leq \infty).$$

The weighted Orlicz space $L_{\Phi,\nu}(0, S)$, $(0 < S \leq \infty)$ consists of all measurable functions $f : (0, S) \to \mathbb{R}$ with finite Luxembourg norm

$$\|f\|_{\Phi,\nu} = \inf \left\{ \lambda > 0 : \int_0^S \Phi\left(\lambda^{-1} |f(t)|\right) \nu(t) dt \leq 1 \right\}. \tag{3.3}$$

Let $\Omega \subset \mathbb{R}^n$ be bounded domain and $T = \mu_n(\Omega)$. A function $f : \Omega \subset (R)^n \to \mathbb{R}$ belongs to Orlicz-Lorentz space $L^*_{\Phi,\nu}(\Omega)$ if $f^* \in L_{\Phi,\nu}(0, T)$ and

$$\|f\|^*_{\Phi,\nu} = \|f^*\|_{\Phi,\nu} = \inf \left\{ \lambda > 0 : \int_0^T \Phi\left(\lambda^{-1} f^*(t)\right) \nu(t) dt \leq 1 \right\}. \tag{3.4}$$

Let $m > 0$ be an integer. The generalized Sobolev-Orlicz space $\dot{W}^m_\Phi(\Omega)$ consists of the set of all functions whose distributional derivatives through order m are contained in $L^*_{\Phi,\nu}(\Omega)$. That is,

$$\dot{W}^m_\Phi(\Omega) = \left\{ f \in L^*_{\Phi,\nu}(\Omega) : \|D^\alpha f\|_{L^*_{\Phi,\nu}(\Omega)} < \infty, |\alpha| \leq m \right\}. \tag{3.5}$$

4 Equivalent description of the associated space to the cone of decreasing functions from Orlicz space.

We consider here the description of the dual space for the cone of all nonnegative decreasing functions from Orlicz space without any priory restrictions on Young function and weight function.

Introduce a (quasi)normed cone

$$M_\Phi := \left\{ g \in L_{\Phi,\nu}(0, T) : 0 \leq g \downarrow ; g(+0) = \lim_{t \to +0} g(t) < \infty \right\} \tag{4.1}$$

of non negative decreasing functions from Orlicz space $L_{\Phi,\nu}(0, T)$ with norm

$$\|g\|_{M_\Phi} := \|g\|_{L_{\Phi,\nu}(0,T)}.$$

The associate RIS M'_Φ for the cone M_Φ consists of those measurable functions such that

$$\|g\|_{M'_\Phi} = \sup \left\{ \int_0^S g^* f dt : f \in M_\Phi, \|f\|_{M_\Phi} \leq 1 \right\} < \infty. \tag{4.2}$$

The main problem is to give sharp estimate for the value in (4.2).

Proposition 4.1. *Let $T < \infty$. For fixed a¿1 put*

$$\rho_a(g;t) = \begin{cases} V(t)^{-1} \int\limits_t^{\beta(t)} g(s)ds, & t \in (0, V^{-1}(a^{-1})) \\ a \int\limits_{V^{-1}(a^{-1})}^{T} g(s)ds & t \in (V^{-1}(a^{-1}), T) \end{cases},$$

where $\beta(t) = V^{-1}(aV(t))$. Then under notations above for measurable function $g \geq 0$

$$\|g\|_{M'_\Phi} \sim \|\rho_a(g;t)\|_{\Psi,\nu} =$$

$$= \inf\left\{\lambda > 0 : \int_0^T \Psi\left(\lambda^{-1}\rho_a(g;t)\right)\nu(t)dt \leq 1\right\}. \tag{4.3}$$

Remark. For different values $a > 1$ norms (4.3) are equivalent.

Proof. We put $1 < b = \sqrt{a}$ and use discretization procedure. Without loss of generality we assume that $V(T) = 1$.

The discrete analogue $\tilde{l}_{\Psi,b}$ of the space $L_{\Psi,\nu}(0,T)$ consists of all sequences $\{a_m\}_0^\infty$ such that

$$\|\{a_m\}\|_{\tilde{l}_{\Phi,b}} = \inf\left\{\lambda > 0 : \sum_{m=o}^{\infty} \Psi\left(\lambda^{-1}|a_m|\right) b^{-m} \leq 1\right\} < \infty. \tag{4.4}$$

We introduce now the discretizing sequence $\{\mu_m\}$ by formulas

$$\mu_m : \quad V(\mu_m) = b^{-m}, \quad m = 0, 1, 2, \ldots \tag{4.5}$$

It is clear that (recall that $V(t) \uparrow$, $V(0) = 0$, $V(T) = 1$)

$$0 < \mu_m \downarrow; \quad \mu_0 = T; \quad \lim_{m\to\infty} \mu_m = 0.$$

The main step for obtaining (4.3) is to prove the discrete version of the answer

$$\|f\|_{M'_\Phi} \sim \left\| \left\{ b^m \int_{\mu_m}^{\mu_{m-1}} f dt \right\}_{m=1}^{\infty} \right\|. \tag{4.6}$$

To see this we put

$$J_\lambda(f) = \int_0^T \Phi\left(\lambda^{-1} f(t)\right)\nu(t)dt. \tag{4.7}$$

By using decreasing sequence (4.5) we may write

$$J_\lambda(f) = \sum_{m=1}^{\infty} \int_{\mu_m}^{\mu_{m-1}} \Phi\left(\lambda^{-1} f(t)\right)\nu(t)dt.$$

Since f is decreasing function and Φ is increasing function we have

$$J_\lambda(f) \leq \sum_{m=1}^{\infty} \Phi\left(\lambda^{-1} f(\mu_m)\right) \int_{\mu_m}^{\mu_{m-1}} \nu(t)dt =$$

$$= \sum_{m=1}^{\infty} \Phi\left(\lambda^{-1} f(\mu_m)\right)\left(V(\mu_{m-1}) - V(\mu_m)\right).$$

According to (4.5) we have

$$V(\mu_{m-1}) - V(\mu_m) = b^{-m+1} - b^{-m} \leq b^{-m+1}, \quad m \in \mathbb{N}$$

and

$$J_\lambda(f) \leq b \sum_{m=1}^{\infty} \Phi\left(\lambda^{-1} f(\mu_m)\right) b^{-m}. \tag{4.8}$$

Analogously we have

$$J_\lambda(f) \geq \sum_{m=1}^{\infty} \Phi\left(\lambda^{-1} f(\mu_{m-1})\right) \int_{\mu_m}^{\mu_{m-1}} \nu(t)dt =$$

$$= \sum_{m=1}^{\infty} \Phi\left(\lambda^{-1} f(\mu_{m-1})\right)\left(b^{-m+1} - b^{-m}\right) \sim \sum_{m=0}^{\infty} \Phi\left(\lambda^{-1} f(\mu_m)\right) b^{-m}. \tag{4.9}$$

Combining results (4.8) and (4.9) we obtain

$$J_\lambda(f) \sim \sum_{m=1}^{\infty} \Phi\left(\lambda^{-1} f(\mu_m)\right) b^{-m}.$$

It means

$$\|f\|_{M_\Phi} \sim \|\{f(\mu_m)\}\|_{\tilde{l}_{\Phi,\nu}}. \tag{4.10}$$

Now we substitute (4.10) in (4.2) and obtain

$$\|g\|_{M'_\Phi} \sim \sup_{0 \leq f \downarrow} \left[\frac{\int_0^\infty gf dx}{\|\{f(\mu_m)\}\|_{\tilde{l}_{\Phi,b}}}\right]. \tag{4.11}$$

The denominator in (4.11) does not depend on values $f(x)$ where $x \neq \mu_m$, $m \in \mathbb{N}$. Therefore we obtain supremum in (4.11) if consider the greatest function with given values $f_0(\mu_m) = f(\mu_m)$. It means that

$$f_0(x) = f(\mu_m) \equiv a_m, \quad x \in [\mu_m, \mu_m - 1), m \in \mathbb{N}.$$

Consequently

$$\|g\|_{M'_\Phi} \sim \sup_{0 \leq a_m \downarrow} \left[\frac{\sum\limits_{m=1}^{\infty} a_m \int_{\mu_m}^{\mu_{m-1}} g dx}{\|\{a_m\}\|_{\tilde{l}_{\Phi,b}}}\right]. \tag{4.12}$$

Now we consider Hardy type operator

$$H[\{\alpha_m\}] = \left\{\sum_{k=1}^{m} \alpha_k\right\}_{m=1}^{\infty}. \tag{4.13}$$

It is easy to see that the Hardy type operator (4.13) is bounded in spaces $l_{1,b}$ and $l_{\infty,b}$. On the other hand the space $l_{\Phi,b}$ is an interpolation space in the pair $(l_{1,b}, l_{\infty,b})$. Therefore operator H is bounded in $l_{\Phi,b}$.

Now for each sequence $\{\alpha_m\} \in \tilde{l}_{\Phi,b}, \alpha_m \geq 0$ consider a sequence $a_m = \sum\limits_{k=1}^{m} \alpha_k$ and we have

$$\|a_m\|_{\tilde{l}_{\Phi,b}} \leq \|H\| \|\{\alpha_m\}\|_{\tilde{l}_{\Phi,b}}. \tag{4.14}$$

It follows from (4.12) and (4.14) that

$$\|g\|_{M'_\Phi} \sim \sup_{0\leq\alpha_m} \left[\frac{\sum\limits_{m=1}^{\infty} \alpha_m \int_{\mu_m}^{\mu_{m-1}} g dx}{\|\{\alpha_m\}\|_{\tilde{l}_{\Phi,b}}} \right] = \left\| \int_{\mu_m}^{\mu_{m-1}} g dx \right\|_{(\tilde{l}_{\Phi,\nu})'}.$$

Now we can apply the principle of duality for discrete weighted Orlicz spaces and we have (4.6).

Now we prove that the right hand side of (4.6) is essentially the same as in (4.3).

Indeed if $g \geq 0$ is measurable function on $(0,T)$ and

$$g'_m \leq g(t) \leq g''_m, \quad t \in [\mu_m, \mu_{m-1}], \quad m \in \mathbb{N}$$

then we can see that

$$c_1 \|\{g'_m\}\|_{\tilde{l}_{\Psi,b}} \leq \|g\|_{L_{\Psi,\nu}} \leq c_2 \|\{g''_m\}\|_{\tilde{l}_{\Psi,b}}.$$

Consequently if $\|\{g'_m\}\|_{\tilde{l}_{\Psi,b}} \sim \|\{g''_m\}\|_{\tilde{l}_{\Psi,b}}$ then $\|\{g'_m\}\|_{\tilde{l}_{\Psi,b}} \sim \|g\|_{L_{\Psi,\nu}}$.

In our case for $t \in [\mu_m, \mu_{m-1}]$, using (4.4) and increasing of V we obtain

$$g'_m \equiv b^{m-1} \int_{\mu_{m-1}}^{\mu_{m-2}} f(s)ds \leq \rho_a(f;t) \leq b^m \int_{\mu_m}^{\mu_{m-3}} f(s)ds \equiv g''_m.$$

Simple estimation shows that

$$\|\{g'_m\}\|_{\tilde{l}_{\Psi,b}} \sim \|\{g''_m\}\|_{\tilde{l}_{\Psi,b}} \sim \left\| \left\{ b^m \int_{\mu_m}^{\mu_{m-1}} f dt \right\}_{m=1}^{\infty} \right\|.$$

Hence completes the proof. ■

Recall that Young's function Φ is said to satisfy the Δ_2 condition if there exists $s_0 > 0$ and $C > 0$ such that

$$\Phi(2s) \leq C\Phi(s) < \infty, \quad s_0 < s < \infty. \tag{4.15}$$

If $\Phi \in \Delta_2$ then we can have more simple description than in Proposition 4.1.

Proposition 4.2. *Suppose that the Young's function Φ of the Orlicz space $L_{\Phi,\nu}$ satisfies the Δ_2 condition. Then under notations above for measurable function $f \geq 0$*

$$\|g\|_{M'_\Phi} \sim \left\| \frac{1}{V(t)} \int_t^T g(s)ds \right\|_{\Psi,\nu}. \tag{4.16}$$

Proof. First, we observe that the equivalence

$$\left\| \left\{ b^m \int_{\mu_m}^{\mu_{m-1}} f dt \right\}_{m=1}^{\infty} \right\| \sim \left\| \left\{ b^m \int_{\mu_m}^{T} f dt \right\}_{m=1}^{\infty} \right\| \tag{4.17}$$

holds if and only if the Hardy type operator

$$G\left[\{a_m\}\right] = \left\{ b^m \sum_{k=1}^{m} b^{-k} a_k \right\} \tag{4.18}$$

is bounded in $\tilde{l}_{\Psi,b}$.

It is easy to show that the operator G is bounded in the spaces $l_{p,b}$ $(p > 1)$ and l_∞ .

If $\Phi \in \Delta_2$ then $\tilde{l}_{\Psi,b}$ is an interpolation space in the couple $(l_{p,b}, l_\infty)$ therefore operator G is also bounded in $\tilde{l}_{\Psi,b}$.

We obtain by analogy with the proof of proposition 4.1

$$\left\| \left\{ b^m \int_{\mu_m}^{S} f dt \right\}_{m=1}^{\infty} \right\| \sim \left\| \frac{1}{V(t)} \int_t^S f(s) ds \right\|_{\Psi,\nu}.$$

Consequently and according to (4.6) we have

$$\|f\|_{M'} \sim \left\| \frac{1}{V(t)} \int_t^S f(s) ds \right\|_{\Psi,\nu}.$$

Hence completes the proof. ■

5 On embedding of Sobolev-Orlicz space in L_∞.

In this section we give the necessary and sufficient condition of embedding of Sobolev-Orlicz space in L_∞.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded domain and $T = \mu_n(\Omega)$. Suppose that the Young's function Φ satisfies the Δ_2-condition. Then the embedding*

$$\dot{W}_\Phi^m(\Omega) \hookrightarrow L_\infty(\Omega) \tag{5.1}$$

is equivalent to the requirement

$$\frac{1}{V(t)} \left(T^{\frac{m}{n}} - t^{\frac{m}{n}} \right) \in L_{\Psi,\nu}(0,T), \tag{5.2}$$

where Ψ is the complementary Young's function of Φ.

Proof. It is known from general theory the embedding (5.1) is equivalent to the requirement [3]

$$t^{\frac{m}{n}-1} \in M'_\Phi(0,T).$$

Now we apply Proposition 4.2 and obtain

$$\|t^{\frac{m}{n}-1}\|_{M'_\Phi} \sim \left\| \frac{1}{V(t)} \int_t^T s^{\frac{m}{n}-1} ds \right\|_{\Psi,\nu} = \left\| \frac{1}{V(t)} \left(T^{\frac{m}{n}} - t^{\frac{m}{n}} \right) \right\|_{\Psi,\nu} < \infty.$$

■

6 Local growth envelope of Sobolev-Orlicz space.

Definition. The local growth envelope function of the Sobolev-Orlicz space is the equivalence class of all positive decreasing functions which are equivalent to

$$E_W(t) := \sup\left\{f^*(t) : f \in \dot{W}^m_\Phi(\Omega), \|f\|_{\dot{W}^m_\Phi(\Omega)} \le 1\right\}.$$

The case that the embedding (5.1) holds is out of consideration here. We want to study in this section the case that $\dot{W}^m_\Phi(\Omega)$ is not continuously embedded in L_∞. We can say, in such case $E_W(t)$ defines a positive, decreasing function in $(0,1]$ which tends to ∞ as t goes to 0.

We denote

$$f_m(t;\tau) = \begin{cases} t^{\frac{m}{n}-1}, & \tau \in [0,t) \\ \tau^{\frac{m}{n}-1}, & \tau \in [t,T) \end{cases}.$$

Then under the condition (5.1) holds following estimate

$$E_W(t) \sim \|f_m(t,\cdot)\|_{M'(0,T)}.$$

According to (4.10)

$$E_W(t) \sim \left\| \frac{1}{V(\tau)} \int_\tau^T f_m(t,s)ds \right\|_{\Psi,\nu}.$$

If $t \in [0,\tau)$ then

$$\int_\tau^T f(t;s)ds = \int_\tau^T s^{\frac{m}{n}-1}ds = T^{\frac{m}{n}} - \tau^{\frac{m}{n}}.$$

In case that $t \in [\tau, T]$

$$\int_\tau^T f(t;s)ds = \int_\tau^t t^{\frac{m}{n}-1}ds + \int_t^T s^{\frac{m}{n}-1}ds = T^{\frac{m}{n}} - \tau t^{\frac{m}{n}-1}.$$

We proved next proposition.

Proposition 6.1. Let $m \in \mathbb{N}$, $m < n$ and

$$\frac{1}{V(t)}\left(T^{\frac{m}{n}} - t^{\frac{m}{n}}\right) \notin L_{\Psi,\nu}(0,T).$$

Then

$$E_W(t) \sim \left\| \frac{1}{V(\tau)} F_m(t,\tau) \right\|_{\Psi,\nu},$$

where

$$F_m(t,\tau) = \begin{cases} T^{\frac{m}{n}} - \tau^{\frac{m}{n}}, & t \in [0,\tau) \\ T^{\frac{m}{n}} - \tau t^{\frac{m}{n}-1}, & t \in [\tau,T] \end{cases}.$$

References.

1. Bennett C.,Sharpley R., *Interpolation of Operators*, Pure and Applied Math. Acad.Press. 1988. Vol. 129,.

2. Antonio M. Caetano; Susanna D. Moura, *Local growth envelopes of spaces of generalized smoothness:the subcritical case,* Math.Nachr. 273, 43–57, 2004.

3. Goldman M. L., *On rearrangement invariant envelopes of the generalized Besov, Sobolev and Calderon spaces,* Proc. of the Int. Conf. dedicated 75th Ann.acad. Reshetnyak Yu.G.,Novosibirsk, 2005.

4. Goldman M.L.; Kerman R., *On the principle of duality in Orlicz-Lorecz spaces,* Proc. of the Int. Conf. "Functional spaces. Differential operators. Problems of mathematical education" dedicated 75th Ann.acad. Kudrjavtsev L.D., 179–183, 1998

5. Davaadulam J., *Application of integral inequalities on the cone of monotone functions for embedding theory of Calderon Spaces,* PhD Thesis, Peoples Friendship University of Russia, 2006.