Stochastic Optimal Control Problem for Life Insurance

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1 Introduction

Beginning in the 1960's, many researchers constructed models to analyze for life insurance and the behavior of investment for an individual under an uncertain lifetime. Yaari (1965) is a starting point for the modern research on the demand for life insurance. Yaari considered the problem of life insurance under an uncertain lifetime for an individual. Following Merton's (1969,1971) work, Richard (1975) used the Yaari's setting for an uncertain lifetime and dynamic programming to consider a life-cycle life insurance and consumption investment problem. Richard employed the dynamic programming technique to attack this problem and to obtain explicit solutions for Constant Relative Risk Aversion (CRRA) case.

2 Formulation of the model

The continuous time economy consists of a financial market and an insurance market. We assume that there is a risk-free security in the financial market whose time-t price is denoted by $S_0(t)$. It evolves according to Dynamic of financial assets for individuals

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt$$

where $S_0(t) = s_0$ is given positive constant and $r(t) : [\widetilde{\tau}, \overline{T}] \to R^+$ is a continuous deterministic function. $\widetilde{\tau}$ is a beginning time to earn wage of a consumer, $\overline{T}$-is a planning horizon for the wage earner. Suppose the wage earner is endowed with the initial wealth $W_{\widetilde{\tau}}$ and will receive the wage at rate $Y(t)$ during the period $[\widetilde{\tau}, \widetilde{T}]$. $\widetilde{T}$ is an age of retire. Here the specified function $Y(t) : [\widetilde{\tau}, \widetilde{T}] \to R^+$ is Borel measurable and satisfies $\int_{\widetilde{\tau}}^{\widetilde{T}} Y(t)dt < \infty$. Define $c(t)$-consumption rate at time t, $P(t)$-amount of insurance purchased at age t, $\theta(t)$-insurance premium rate at time t and $I_{\widetilde{\tau} \leq t \leq \widetilde{T}}$ -indicator function of time set $\widetilde{\tau} \leq t \leq \widetilde{T}$. Given the consumption process c, the wage process $Y(t)$, the premium rate process $\theta(t)P(t)$, the wealth process $W(t)$ for consumer's on $[\widetilde{\tau}, \overline{T}]$ is defined by

$$dW(t) = r(t)W(t)dt + Y(t)I_{\{\widetilde{\tau} \le t \le \widetilde{T}\}}dt - c(t)dt - \theta(t)P(t)dt, \ \widetilde{\tau} \le t \le \overline{T}. \tag{1}$$

We suppose the wage process $Y(t)$ is defined by

$$dY(t) = \mu_y(t)dt + \sigma_y(t)dB(t), \quad \widetilde{\tau} \le t \le \widetilde{T}. \tag{2}$$

Here $\mu_y(t)$ is an average wage function, $\sigma_y(t)$ is a volatility wage function, $B(t)$ is a Brownian motion. Substituting (2) into (1), we have

$$dW(t) = \Big(r(t)W(t) + \mu_y(t)I_{\{\widetilde{\tau} \le t \le \widetilde{T}\}} - c(t) - \theta(t)P(t)\Big)\, dt + \sigma_y(t)I_{\{\widetilde{\tau} \le t \le \widetilde{T}\}}dB(t). \tag{3}$$

2.1 Objective function

We denoted lifetime t, a nonnegative random variable defined on the probability space $\{\Omega, F, P\}$. Now suppose that the random variable τ has a probability distribution with underlying probability density function $\pi(t)$ and distribution function given by

$$F(t) = P(\tau < t) = \int_0^t \pi(t)dt.$$

The function $S(t)$, which is called the survivor function, is defined to be the probability that the survival time is greater than or equal to t

$$S(t) = P(t \le \tau \le \overline{T}) = P(\tau \le \overline{T}) - P(\tau < t) = 1 - F(t) = \int_t^{\overline{T}} \pi(t)dt,$$

where $S(0) = 1, \ S(\overline{T}) = 0$, and $S^{'}(t) = -\pi(t) < 0$.
The hazard function represents the instantaneous death rate for wage earner who has survived to time t and it is defined by

$$\lambda(t) = \lim_{\delta t \to 0} \frac{P(t \le \tau \le t + \delta t | \tau \ge t)}{\delta t}.$$

From this definition we can obtain some useful relationships between $\pi(t)$ and $\lambda(t)$, namely

$$\lambda(t) = \frac{\pi(t)}{S(t)}. \tag{4}$$

It then follows that

$$\lambda(t) = \frac{\pi(t)}{S(t)} = -\frac{S'(t)}{S(t)} = -\frac{d}{dt}\left(\ln S(t)\right).$$

The case in which the survivor function is given by

$$S(t) = e^{-\int_0^t \lambda(\tau)d\tau}.$$

Given an uncertain lifetime t with a density function $\pi(t)$, consumer's expected utility is

$$E_\pi \psi(t) = \int_0^{\overline{T}} \pi(t)\psi(t)dt = \int_t^{\overline{T}} \left[\pi(t) \cdot \int_0^t e^{-\int_0^\tau \rho(v)dv} U\left(\tau, c(\tau)\right) d\tau + \Phi\left(t, Z(t)\right)\right] dt$$

$$= \int_0^{\overline{T}} \left(\pi(t) \cdot \int_0^t e^{-\int_0^\tau \rho(v)dv} U(c(\tau))d\tau \right) dt + \int_0^{\overline{T}} \pi(t) \cdot \Phi\left(t, Z(t)\right) dt.$$

Where $U\left(\tau c(\tau)\right)$ is the utility for consumption, $\Phi(t, Z(t))$ is the utility for the bequest, $e^{-\int_0^\tau \rho(v)dv}$ is the discount function. Using integration by parts as before, we can obtain objective function as

$$E_\pi \psi(t) = \int_0^{\overline{T}} \left[\pi(t)\Phi\left(t, Z(t)\right) + S(t) e^{-\int_0^t \rho(\tau)d\tau} U\left(t, c(t)\right) \right] dt.$$

2.2 Life insurance model

We suppose that problem for life insurance can be written bellow

$$\max_{\{c(t),\ P(t)\}} \left\{ \int_0^{\overline{T}} \left[\pi(t)\Phi(t, Z(t)) + S(t) e^{-\int_0^t \rho(\tau)d\tau} U(t, c(t)) \right] dt \right\}.$$

Subject to:

$$\begin{aligned} dW(t) &= \left(r(t)W(t) + \mu_y(t) I_{\left\{\widetilde{\tau} \le t \le \widetilde{T}\right\}} - c(t) - \theta(t)P(t) \right) dt + \sigma_y(t) I_{\left\{\widetilde{\tau} \le t \le \widetilde{T}\right\}} dB(t), \\ Z(t) &= W(t) + P(t), \quad W(\widetilde{\tau}) = W_{\widetilde{\tau}}, \end{aligned} \tag{5}$$

where $Z(t)$ is the contingent bequest.

3 Explicity solution's necessary condition

Let the present value of the indirect utility function $J(t, W)$ at time t be

$$J(t, W) = \max_{\{c(t),\ P(t)\}} \left\{ \int_t^{\overline{T}} \left[\pi(t)\Phi(t, Z(t)) + S(t) e^{-\int_0^t \rho(\tau)d\tau} U(t, c(t)) \right] dt \right\}.$$

Now we can write HJB equation for problems (5). HJB equation is

$$\begin{aligned} 0 = \max_{\{c(t),\ P(t)\}} & \Big\{ \lambda(t)S(t)\Phi(t, Z(t)) + S(t) e^{-\int_0^t \rho(\tau)d\tau} U(t, c(t)) + J_t \\ & + \left(r(t)W(t) + \mu_y(t) I_{\left\{\widetilde{\tau} \le t \le \widetilde{T}\right\}} - c(t) - \theta(t)P(t) \right) J_W \\ & + \frac{1}{2} \sigma_y^2(t) I_{\left\{\widetilde{\tau} \le t \le \widetilde{T}\right\}} J_{WW} \Big\} \end{aligned} \tag{6}$$

Where J_t, J_W, and J_{WW} are partial derivatives.

For the simplicity, let the value function be of the forms

$$J(t, W) = S(t) e^{-\int_0^t \rho(\tau)d\tau} F(t, W) = e^{-\int_0^t (\rho(\tau)+\lambda(\tau))d\tau} F(t, W),$$

$$\Phi(t, Z(t)) = \frac{1}{\lambda(\mathrm{t})} \varphi(Z, t) e^{-\int_0^t \rho(\tau)d\tau}.$$

For $F(t,W)$ function, the above equation is

$$0 = \max_{\{c(t),\ P(t)\}} \{\varphi(Z,t) + U(t,c(t)) - (\rho(t)+\lambda(t))F + F_t + \left(r(t)W(t) + \mu_y(t)I_{\{\widetilde{\tau}\le t\le \widetilde{T}\}} - c(t) - \theta(t)P(t)\right)F_W + \frac{1}{2}\sigma_y^2(t)I_{\{\widetilde{\tau}\le t\le \widetilde{T}\}}F_{WW}\Big\}.$$

Assuming that $\rho(t)+\lambda(t) = \rho^*(t)$, $\vartheta(t) = \frac{1}{t-\widetilde{\tau}}$ in the above equation, we have

$$0 = \max_{\{c(t),\ P(t)\}} \{\varphi(Z,t) + U(t,c(t)) - \rho^*(t)F + F_t + \left(r(t)W(t) + \mu_y(t)I_{\{\widetilde{\tau}\le t\le \widetilde{T}\}} - c(t) - \frac{1}{t-\widetilde{\tau}}P(t)\right)F_W + \frac{1}{2}\sigma_y^2(t)I_{\{\widetilde{\tau}\le t\le \widetilde{T}\}}F_{WW}\Big\}. \quad (7)$$

Hence the first order conditions are

$$\begin{cases} U^{'}(c) = F_W \\ \varphi_Z = \frac{1}{t-\widetilde{\tau}}F_W. \end{cases} \quad (8)$$

4 Main result

Theorem 1. *In the case the utility function for consumption is of the form CARA,* $U(t,c) = -\frac{\alpha(t)}{\beta}e^{-\beta c}$ *, and the utility function for bequest is of the form* $\Phi(t,Z(t)) = -A(t)\frac{1}{\lambda(t)}e^{-\int_0^t \rho(\tau)d\tau}e^{-K(t)Z}$*, and* $F(t,W) = -b(t)e^{-a(t)W}$ *then the solution of the problem (5) and its parameters are determined by followings:*

Optimal controls are:

- $c = \frac{1}{\beta}\left(a(t)W + \ln\frac{\alpha(t)}{a(t)b(t)}\right), \qquad P(t) = \left(\frac{a(t)}{K(t)} - 1\right)W$

  and parameters are:

- $a(t) = \left[exp\left\{\int_{\overline{T}}^t (r(\tau) + \frac{1}{\tau-\overline{\tau}})d\tau\right\}\cdot \left(\pi(\overline{T})\cdot K(\widetilde{T})\cdot(\overline{T}-\widetilde{\tau}) - \int_{\overline{T}}^t\left(\frac{1}{\beta} + \frac{1}{K(t)\cdot(\tau-\widetilde{\tau})}\right)exp\left\{-\int_{\overline{T}}^{\tau}\left(r(s)+\frac{1}{s-\widetilde{\tau}}\right)ds\right\}d\tau\right)\right]^{-1}$

- $b(t) = exp\left\{\left(\int_{\overline{T}}^t exp\left\{\frac{a(\tau)}{\beta}\right\}d\tau\right)\left(\frac{A(\overline{T})}{\pi(\overline{T})} + \int_{\overline{T}}^t D(\tau)\left(-\int_{\overline{T}}^s exp\left\{\frac{a(s)}{\beta}\right\}ds\right)d\tau\right)\right\}$ *, where* $A(t) = \frac{a(t)b(t)}{K(t)\cdot(t-\widetilde{\tau})}$*,*

$$D(t) = \left(\mu_y(t)I_{\{\widetilde{\tau}\le t\le \widetilde{T}\}} - \frac{1}{K(t)\cdot(t-\widetilde{\tau})} - \frac{1}{\beta}\left(1 + ln\frac{\alpha(t)}{a(t)}\right) - \frac{1}{2}\sigma_y^2(t)I_{\{\widetilde{\tau}\le t\le \widetilde{T}\}}a(t)\right)a(t) + \rho^*(t).$$

Proof. We shall guess the value function F to be of the form

$$F(t,W) = -b(t)e^{-a(t)W}. \tag{9}$$

Here $a(t)$, $b(t)$ are deterministic functions. By using the first equation of the first-order condition (8), we have

$$\alpha(t)e^{-\beta c} = a(t)b(t)e^{-a(t)W}.$$

This implies

$$c = \frac{1}{\beta}\left(a(t)W + \ln\frac{\alpha(t)}{a(t)b(t)}\right). \tag{10}$$

Similarly by using the second equation of the first-order condition (8), we have

$$\varphi(Z,t) = -A(t)e^{-K(t)Z}.$$

It follows

$$\varphi_{\mathrm{Z}} = A(t)K(t)e^{-K(t)Z}.$$

Consequently we get

$$A(t)K(t)e^{-K(W+P)} = \frac{a(t)b(t)}{t-\widetilde{\tau}}e^{-a(t)W},$$

and we have

$$A(t) = \frac{a(t)b(t)}{K(t)\cdot(t-\widetilde{\tau})}, \quad P = (\frac{a(t)}{K(t)} - 1)W. \tag{11}$$

Substituting these results and F_W, F_{WW} into the HJB equation (7), we have obtained the following equation

$$\begin{aligned} 0 = &- \frac{a(t)b(t)}{K(t)\cdot(t-\widetilde{\tau})} - \frac{a(t)b(t)}{\beta} + \rho^*(t)b(t) + \left(b(t)a^{'}(t)W - b^{'}(t)\right) \\ &+ \left(r(t)W(t) + \mu_y(t)I_{\{\widetilde{\tau}\le t\le \widetilde{T}\}} - \frac{1}{\beta}\left(a(t)W + ln\frac{a(t)}{a(t)b(t)}\right) - \frac{1}{t-\widetilde{\tau}}\left(\frac{a(t)}{K(t)} - 1\right)W\right)a(t)b(t) \\ &- \frac{1}{2}\sigma_y^2 I_{\{\widetilde{\tau}\le t\le \widetilde{T}\}}\left(a(t)\right)^2 b(t). \end{aligned}$$

Hence we can find the Bernoulli equation for $a(t)$

$$a^{'}(t) + \left(r(t) + \frac{1}{t-\widetilde{\tau}}\right)a(t) = \left(\frac{1}{\beta} + \frac{1}{K(t)\cdot(t-\widetilde{\tau})}\right)a^2(t).$$

It follows

$$\begin{aligned} a(t) = &\left[exp\left\{\int_{\widetilde{T}}^{t}(r(\tau) + \frac{1}{\tau-\widetilde{\tau}})d\tau\right\}\times\right. \\ &\left.\left(const_A - \int_{\widetilde{T}}^{t}\left(\frac{1}{\beta} + \frac{1}{K(t)\cdot(\tau-\widetilde{\tau})}\right)exp\left\{-\int_{\widetilde{T}}^{\tau}\left(r(s) + \frac{1}{S-\widetilde{\tau}}\right)ds\right\}d\tau\right)\right]^{-1}, \end{aligned}$$

and for $b(t)$

$$(\ln b(t))^{'} - \frac{a(t)}{\beta}\ln b(t) = D(t). \tag{12}$$

Here

$$D(t) = \left(\mu_y(t)I_{\{\widetilde{\tau}\leq t\leq\widetilde{T}\}} - \frac{1}{K(t)\cdot(t-\widetilde{\tau})} - \right.$$
$$\left. -\frac{1}{\beta}\left(1 + ln\frac{\alpha(t)}{a(t)}\right) - \frac{1}{2}\sigma_y^2(t)I_{\{\widetilde{\tau}\leq t\leq\widetilde{T}\}}a(t)\right)a(t) + \rho^*(t).$$

From (12), we have

$$b(t) = exp\left\{\left(\int_{\widetilde{T}}^{t} exp\left\{\frac{a(\tau)}{\beta}\right\}d\tau\right)\left(const_B + \int_{\widetilde{T}}^{t} D(\tau)\left(-\int_{\widetilde{T}}^{s} exp\left\{\frac{a(s)}{\beta}\right\}ds\right)d\tau\right)\right\}.$$

Using the boundary condition, we can get

$$\text{const}_A = \pi(\widetilde{T})\cdot K(\widetilde{T})\cdot(\widetilde{T}-\widetilde{\tau}),$$

and

$$\text{const}_B = \frac{A(\widetilde{T})}{\pi(\widetilde{T})}$$

which completes the proof. □

5 CONCLUSION

In this paper, we set up a new model to investigate the problem of optimal life insurance purchase, consumption for stochastic wage under an uncertain lifetime.

An analytical solution of above portfolio optimization problem was investigated.

Following mentioned result has been obtained by assuming the utility function to be of the form CARA: optimal consumption and optimal insurance premium are linear in wealth with time variable coefficient.

6 REFERENCE

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