A generalization of the concept of Toeplitz graphs*

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Abstract

A suitable generalization for both the concept of Cayley graphs and that of Toeplitz graphs is given in this note and a number of interesting open problems are proposed. A natural decomposition theorem is obtained for generalized Topelitz graphs and connected generalized Toeplitz graphs. These are new observations and their proofs are direct from the definition.

Keywords: Cayley graphs, generalization, Toeplitz graphs, Toeplitz matrices

1 Toeplitz graphs

The concept of a Toeplitz graph was a natural combinatorial rendering of the concept of a Toeplitz matrix. A *Toeplitz matrix*, introduced first by Otto Toeplitz (see [10, Chapter 20]), is a square matrix in which entries in every diagonal parallel to the main diagonal are equal, namely an $n \times n$ matrix $T = (t_{ij})$ with the property that $t_{i+1\,j+1} = t_{ij}$ for all $1 \leq i, j \leq n-1$. An example of a Toeplitz matrix is shown below.

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$$\begin{pmatrix} 0 & 1 & 0 & 2 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & -2 \\ 3 & 0 & 1 & 0 & 1 & 0 & 2 \\ -1 & 3 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 3 & 0 & 1 & 0 \end{pmatrix}$$

A Toeplitz matrix is determined by its $2n-1$ leading entries of rows and columns. For a Toeplitz matrix T, there are two ways a binary matrix may be obtained from T: one is by taking a binary matrix A whose entries are the indicator binary values of the corresponding entries of T, and the other is where the entries are binary variables giving the parities of the corresponding entries of T. In either case a finite (directed) graph may be determined with the resulting matrix as the adjacency matrix.

The indicator binary matrix and the parity binary matrix of this Toeplitz matrix are shown below. Note that the indicator binary matrix just replaces each nonzero entry by 1, and the parity binary matrix replaces each entry by its parity which means that if t_{ij} is even it is replace by 0 and by 1 otherwise.

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The first of these two matrices is symmetric, although it was not obtained from a symmetric Toeplitz matrix arising as indicator binary matrix. This is a coincidence. Denote this matrix by A.

Let A be a symmetric binary Toeplitz matrix with entries in the main diagonal all 0. Then A is determined by its $n-1$ column leaders above the main diagonal. This is a binary sequence of length n. This sequence determines all information on all graph invariants but we do not know how. Denote by $a_1, a_2, \cdots, a_r$ the column numbers with leading entry 1. Then the matrix A may be determined by this sequence of positive integers, and the Toeplitz graph may be denoted by $T_n(a_1, a_2, \cdots, a_r)$. This simply means that if $V = \{1, 2, \cdots, n\}$ then the undirected Toeplitz graph $T_n(a_1, a_2, \cdots, a_r)$ has vertex set V such that a vertex i is adjacent to vertices $i + a_j$, $j = 1, 2, \cdots, r$ for $i + a_j \leq n$.

The Toeplitz graph with adjacency matrix A is shown in Figure 1.

Let $A = \{a_1, a_2, \cdots, a_r\} \subset \mathbb{N}$ be a set of positive integers, with $a_1 < a_2 < \cdots < a_r < n$. A finite (undirected) *Toeplitz graph* $T_n(A)$ is a graph with

$$V = \{1, 2, \cdots, n\},\ E = \{ij : |i-j| \in A\}.$$

If $|V|$ is infinite then the Toeplitz graph is infinite. For each Toeplitz graph T there exists a circulant graph (Cayley graph of a cyclic group) such that T is a spanning subgraph of the circulant graph.

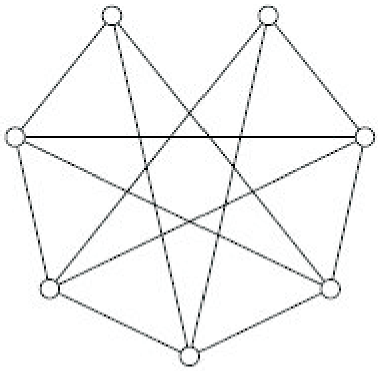

Fig. 1: The Toeplitz graph $T_7(1, 3, 4)$.

2 A generalization

Let S be a semigroup and $A \subset V \subseteq S$ with $|V|$ finite. The *generalized Toeplitz* graph $G = T_V(A)$ is defined to have

$$V(G) = V,\ E(G) = \{(v, av) : a \in A,\ av \in V\}.$$

In addition, if condition

$$1 \notin A,\ A^{-1} = A$$

is imposed then an edge (v, av) as an ordered tuple becomes a set $\{v, av\}$ and the graph is simple and undirected. Conditions for $T_V(A)$ to be undirected was investigated in [12]. Note that in [12, 17], the fact that this also generalizes the concept of Toeplitz graphs was not noticed. The intention of these papers was to generalize Cayley graphs. Our definition here is also more general than that in [12, 17], for if $V = S$ and S is finite then our definition becomes that of [12, 17].

Now for $S = \mathbb{N}$, $T_V(A)$ is a classical Toeplitz graph. Whereas the family of classical Toeplitz graphs does not contain all Cayley graphs as subfamily, the family of generalized Toeplitz graphs contains the family of all Cayley graphs as subfamily, where S is a finite group, $V = S$ and the set A satisfies the additional condition: $1 \notin A$ and $A^{-1} = A$ (or, unitity-free and self-inverse). Hence the concept of a generalized Toeplitz graph is a common generalization of both Toeplitz graphs and Cayley graphs of semigroups.

This generalization covers a variety of additional families, in addition to Cayley graphs and Toeplitz graphs. These will be illustrated by examples.

Example 1. **Toeplitz graphs**. Let $V = \{1, 2, \cdots, n\}$, $A = \{a_1, a_2, \cdots, a_r\}$ with $1 \le a_1 < a_2 < \cdots < a_r < n$. Then the Toeplitz graph $T_n(A)$ is just the generalized Toeplitz graph $T_V(A)$ with $S = \mathbb{N}$.

Example 2. **Cayley graphs**. Let S be a finite semigroup, $A \subset S$ not contain a left unity. Then the Cayley graph $G = \mathrm{Cay}(S, A)$ is a generalized Toeplitz graph.

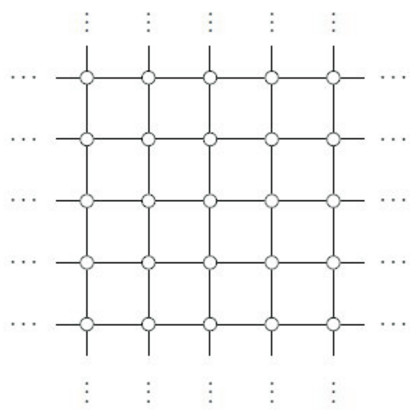

Fig. 2: The 2-dimensional grid.

Example 3. **Grids and finite grids**. For $d \geq 1$ let

$$V = S = \mathbb{Z}^n,\ A = \{(1,0,\cdots,0),(0,1,\cdots,0),\cdots,(0,0,\cdots,1)\}.$$

Then $T_V(A)$ is the n-dimensional grid graph. The case for $n = 2$ is illustrated in Figure 2.

For finite grids, let $S = \mathbb{Z}^n$, $V = \{(v_1, v_2, \cdots, v_n) : \forall i, 0 \leq v_i \leq b_i\}$ and $A = \{(1,0,\cdots,0),(0,1,\cdots,0),\cdots,(0,0,\cdots,1)\}$. Then

$$T_V(A) \simeq P_{b_1} \Box P_{b_2} \Box \cdots \Box P_{b_m}$$

the finite grid box graph, which is the direct product of paths $P_{b_1}, \cdots, P_{b_n}$.

Example 4. **Vector semigroups**. Let S be a vector semigroup, for example

$$S = \{(x,y) : ax \leq y \leq bx,\ a \leq b, x \geq 0\}$$

under vector addition. Let $A, V \subset X$ with $1 \notin A$ and $|A|$ finite. Then $T_V(A)$ is a generalized Toeplitz graph. Figure 2 illustrates an example with $a = 1$, $b = 3$, $V = \{(x,y) : x, y \in \mathbb{N},\ 1 \leq x \leq y \leq 7,\ y \leq 3x\}$ and $A = \{(1,0),(0,1)\}$. It is easy to see that $|V| = 17$. This example also shows that there are generalized Toeplitz graph which are neither Cayley graphs nor Toeplitz graphs.

3 A decomposition

In this section an observations on decomposition will be proved, proofs of which are direct from definition.

Lemma 4. *Let S be a semigroup, $A, B, V \subset S$ with $1 \notin A \cup B$ and $|A \cup B|$ finite. Then*

$$T_V(A \cup B) = T_V(A) \cup T_V(B).$$

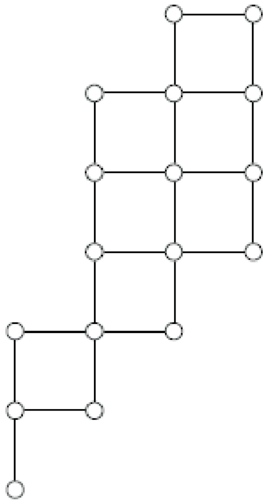

Fig. 3: Generalized Toeplitz graph of a vector semigroup.

It is clear that $V(T_V(A\cup B)) = V = V(T_V(A)\cup T_V(B))$. Now for each $x, y \in V$,

$$xy \in E(T_V(A\cup B)) \Leftrightarrow \exists a \in A, y = xa \text{ or } \exists b \in B, y = xb \Leftrightarrow xy \in E(T_V(A)\cup T_V(B)).$$

Hence $T_V(A\cup B) = T_V(A)\cup T_V(B)$. □

Let $G = T_V(A)$ be a connected generalized Toeplitz graph. If G has no proper connected generalized Toeplitz factor, then G cannot be decomposed into more than one proper connected generalized Toeplitz factors. Examples of connected generalized Toeplitz graph that may not be decomposed into proper connected generalized Toeplitz factors are paths and cycles. If G cannot be decomposed into proper connected generalized Toeplitz factors, then G is may be thought of as decomposed into itself as just one factor G, since G is a factor of G. Suppose that for each connected generalized Toeplitz graph H with $V(H) = V(G)$ and $\|H\| < \|G\|$, H is decomposed into H or proper connected generalized Toeplitz factors. Let H be a proper connected generalized Toeplitz factor of G such that $J = G\backslash E(H)$ is connected. Then there exists $B \subset A$ such that $H = T_V(B)$. Then $V(H) = V(G) = V(J)$ and $\|H\|, \|J\| < \|G\|$. Now $A\backslash B \subset A$ and $J = T_V(A\backslash B)$ is a connected generalized Toeplitz graph. If H and J are not decomposable into proper connected generalized Toeplitz factors then clearly $G = H\cup J$ with $E(H)\cap E(J) = \emptyset$ and this is a decomposation of G into proper connected generalized Toeplitz factors H and J. If H or J may be further decomposed into proper connected generalized Toeplitz factors then so may be G. Hence we have proved

Theorem 3.1. *Let G be a connected generalized Toeplitz graph. Then G has a decomposition into connected generalized Toeplitz factors.* □

4 Intermediate value theorems

In this section, I discuss possible further research problems.

A graph is *hamiltonian* if it has a spanning cycle.

Problem 1. Characterize generalized Toeplitz graphs which are hamiltonian.

Apart from hamiltonicity, existence of long cycles or cycles of certain lengths in a connected generalized Toeplitz graphs may be interesting. Decompositions into subgraphs with certain property is also interesting.

Problem 2. What condition implies that a connected generalized Toeplitz graph may be decomposed into cycles? Equivalently, which connected generalized Toeplitz graphs are Eulerian?

Concerning invariants of graphs, more specifically, graph parameters, there are a rich repository of interesting problems.

Let $\mathscr{G}$ be a family of graphs and S be a set with an equivalence relation $\sim$ defined on it. A mapping $f : \mathscr{G} \to S$ is called an *invariant* of graphs if for each $G, H \in \mathscr{G}$, $G \simeq H \Rightarrow f(G) \sim f(H)$. Examples of graph invariants are chromatic and Tutte polynomials of graphs, spectra and Laplacian spectra of graphs, autormorphism group, and the degree sequence. For S a set of numbers or a set of sequences of numbers, the equivalence relation $\sim$ is usually just the equal sign $=$. In terms of mappings, a function taking its argument a graph G is an invariant if for each automorphism φ of G, $f(\varphi(G)) = f(G)$, or simply, $f\varphi = f$, as the above is true for all graphs G in the family. If $S = \mathbb{F}$ an ordered field and the equivalence relation $\sim$ is equality $=$, then f is called a *graph parameter*.

Let $f : \mathscr{G} \to \mathbb{F}$ be a graph parameter, where $\mathbb{F}$ is an ordered field. A parameter f is said to satisfy the *intermediate value theorem*, if for each $k \in \mathbb{F}$ satisfying $f_{\text{min}} \le k \le f_{\text{max}}$, there exists a graph $G \in \mathscr{G}$ with $|G| = n$ and $f(G) = k$. A parameter f is said to satisfy the *absolute intermediate value theorem*, if for each $k \in \mathbb{F}$ satisfying $f_{\text{min}} \le k \le f_{\text{max}}$, there exists a graph $G \in \mathscr{G}$ with $f(G) = k$. It is clear that the intermediate value property implies the absolute intermediate value property.

Recall from [5, page 126] that the *independence number* of a graph G is the cardinality of a maximum independent set in G, that is a set S of vertices with largest possible cardinality such that for each $x, y \in S$, $xy \notin E(G)$. We ask whether the independence number of generalized Toeplitz graphs of order n satisfies the intermediate value property (absolute intermediate value property).

Problem 3. Let $\mathscr{T}_n$ be the family of connected generalized Toeplitz graphs of order n. Let $\alpha(G)$ be the independence number of graph G. Is it true that for each $k \in \mathbb{N}$ with $\alpha_{\min} \le k \le \alpha_{\max}$, there exists a connected generalized Toeplitz graph $G \in \mathscr{T}_n$ such that $\alpha(G) = k$?

Let G be a graph and $S \subseteq V(G)$. If for each cycle $C \subseteq G$, $S \cap V(C) \neq \emptyset$, then S is called a *decycling set* (or a *feedback set*) of G. The cardinality of a minimum decycling set of G is called the *decycling number* of G and this parameter is denoted $\phi(G)$. The reader is referred to [2] for a survey of results on decycling numbers of graphs. Does the decycling number of connected generalized Toeplitz graphs of order n satisfy the intermediate value property?

Problem 4. Let $\mathscr{T}_n$ be the family of connected generalized Toeplitz graphs of order n. Let

$\phi(G)$ be the decycling number of graph G. Is it true that for each $k \in \mathbb{N}$ with $\phi_{\min} \leq k \leq \phi_{\max}$, there exists a connected generalized Toeplitz graph $G \in \mathscr{T}_n$ such that $\phi(G) = k$?

These are just two examples of questions concerning a graph parameter f. Other examples of graph parameters are the chromatic number, the chromatic index (or edge chromatic number), the clique number, the cover number, and the connectivity and many of its variations.

A necessary and sufficient condition for a Toeplitz graph to be connected is known [11]. The reader may also see [15, Theorem 1.1, page 3].

Problem 5. Find a necessary and sufficient condition for a generalized Toeplitz graph to be connected.

5 Some remarks

As this generalization of the concept of Toeplitz graphs is new and it is an appropriate generalization, many interesting problems remain open, as is for any new direction.

These problems fall into several types. The first type of questions are whether results obtained for Toeplitz graphs or for Cayley graphs may be obtained for generalized Toeplitz graphs. For literature on Cayley graphs the reader is referred to [1, 3, 8, 9, 13, 16], and for literature on Toeplitz graphs the reader is referred to [4, 6, 7, 11, 14].

The second type of questions include some of the problems that have been proposed in this note, and exclude also some. These are new questions for generalized Toeplitz graphs, questions that have not been asked about Cayley graphs or Toeplitz graphs. These may include study of new parameters or invariants for generalized Toeplitz graphs. By attempting these questions, one might hope to bypass difficulties encountered previously in special cases and obtain new result which are more general.

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