Normalized mathematical models with finished products of biological origin

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Abstract

In this work, we consider and investigate some two dimensional ecol-economical model, where the right end of dynamic system coincides with the turnpike point. Our goal is to reach the turnpike point as soon as possible, where the original problem is separated into the time optimal problem and the mathematical programming problem for finding the turnpike point.

Statements and discussion

We assume that a cattle-farm has a productive sector defining it's capital-labor ratio $k(t)$ i.e. $k(t) = \frac{K(t)}{L}, Y(t) = AK^{\alpha}(t)L^{1-\alpha}, 0 < \alpha < 1, L$-constant, where $Y(K, L) = AK^{\alpha}L^{1-\alpha}$-production function, L-Labor, K-capital.
Suppose that $s - th(0 < s < 1)$ part of total product Y is used for investment, the other for stock nomadic breeding. Let ρ be the coefficient of efficiency of k and c_1 be the total input for one animal. Then ecol-economical strength of farm may be expressed as follows

$$\varphi(t) = \frac{cx(t)}{\rho(1 - s(t))Ak^{\alpha}(t)} = \frac{cx(t)L}{\rho(1 - s(t))AY(t)} \tag{1}$$

and dynamics of $K(t)$ and $k(t)$ are described, respectively

$$\dot{K}(t) = s(t)AK^{\alpha}(t)L^{1-\alpha} - \mu K(t),$$
$$\dot{k}(t) = s(t)Ak^{\alpha}(t) - \mu k(t), \tag{2}$$

where μ is the coefficient of capital amortization.
In our work [6], we constructed a ecol-economical model with finished products of biological

origin, where function $g(x, \varphi)$ denotes the rate of growth of herd population. There, we proved the existence of stationary regime for sufficiently long time period $[0, T]$. Next, we will use next the function $g(x, \varphi)$ for reproductive process.

$$g(x(t), \varphi(t)) = ax(t)(1 - \varphi(t)) = ax(t)(1 - \frac{cx(t)}{\rho(1 - s(t))Ak^{\alpha}(t)}), \tag{3}$$

where $x(t)$-number of animals, $a = g'_x(0, 0)$-maximal possible rate of number growth. By virtue of above assumptions we can describe the following model

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T (u(t)x(t) + Wx(t))dt \to \max, \tag{4}$$

$$\dot{x}(t) = ax(t)(1 - \frac{cx(t)}{\rho(1 - s(t))Ak^{\alpha}(t)}) - u(t)x(t), \tag{5}$$

$$\dot{k}(t) = s(t)Ak^{\alpha}(t) - \mu k(t), \tag{6}$$

$$k(0) = k^0 > 0, x(0) = x^0, \tag{7}$$

$$0 \le u(t) \le \bar{u}, 0 \le s(t) \le 1, -a < W < a, \bar{u} > a, \tag{8}$$

where W is the profit from one animal, $u(t)$ is the slaughter intensity of cattle at moment $t \subseteq [0, T]$. This closed eco-economical model has two control functions $u(t)$ and $s(t)$. Our goal is to find optimal controls maximizing the integral (4).

2. Computation of the optimal stationary regime

It is known in [2-5] that for sufficiently long time period $[0, T]$ in the model $(9) - (12)$ of economical growth there exists the turnpike optimal regime $k^* = (\frac{\alpha A}{\mu})^{\frac{1}{1-\alpha}}$ with corresponding a constant control $s = \alpha$ in interval $[T_1 T_2] \subseteq [0T]$.

$$\int_0^T (1 - s(t))k^{\alpha}(t)dt \to \max, \tag{9}$$

$$\dot{k}(t) = s(t)Ak^{\alpha}(t) - \mu k(t), \tag{10}$$

$$k(0) = k^0 > 0, k(T) = k^T > 0, \tag{11}$$

$$0 \le s(t) \le 1 \quad . \tag{12}$$

Moreover $\lim\limits_{T \to \infty} \frac{T_2 - T_1}{T} = 1$ and T_1, T_2 depend only on $k^0, k^*.T$.

On the other hand, we proved in [1,6] that for sufficiently long time period $[0, T]$ and fixed s, A, k in the ecological model $(14) - (16)$ there exists the optimal turnpike regime

$$x^*_W = \frac{(a + W)\rho(1 - s)Ak^{\alpha}}{2ac}, \tag{13}$$

with corresponding a optimal control $\frac{a-W}{2}$ in interval $[T^*_1, T^*_2] \subseteq [0T]$.

$$\int_0^T (u(t)x(t) + Wx(t))dt \to \max, \tag{14}$$

$$\dot{x}(t) = ax(t)(1 - \frac{cx(t)}{\rho(1-s(t))Ak^{\alpha}(t)}) - u(t)x(t) \tag{15}$$

$$x(0) = x^0, 0 \le u(t) \le \bar{u}, -a < W < a, \bar{u} > a \quad . \tag{16}$$

Numbers T_1^*, T_2^* depends only on x^0, T and satisfy the condition $\lim\limits_{T\to\infty} \frac{T_2^*-T_1^*}{T} = 1$.
Proposition 1. In problem $(4)-(8)$, there exists the optimal stationary process

$$\begin{array}{l} k^* = (\frac{\alpha A}{\mu})^{\frac{1}{1-\alpha}}, x^* = \frac{(a+W)\rho(1-s^*)(k^*)^{\alpha}A}{2ac}, \\ s^* = \alpha, u^* = \frac{a-W}{2} \end{array} \tag{17}$$

on $[T^*, \infty]$, which presents the turnpike point, where $T^* \ge max(T_1, T_1^*)$.
Proof. The function $F(x,u) = (u+W)x$ for given $0 < s < 1-\varepsilon$ and under constraints $a(1 - \frac{cx}{\rho(1-s)k^{\alpha}A}) = u, 0 \le u \le \bar{u}$ reaches the maximal value of F^*, when

$$x^* = \frac{(a+W)\rho(1-s)k^{\alpha}A}{2ac}, u^* = \frac{a-W}{2}.$$

Therefore,

$$F^* = \frac{(a+W)^2\rho(1-s)k^{\alpha}A}{4ac}. \tag{18}$$

Conversely, the right hand of the expression (13) reaches own value under the conditions $0 < s < 1, k = (\frac{sA}{\mu})^{\frac{1}{1-\alpha}}$, if and only if
$s^* = \alpha, k^* = (\frac{\alpha A}{\mu})^{\frac{1}{1-\alpha}}$. The proposition 1 is proved.

§3. Investigation of time-optimal problem

In this part, we will consider the problem of fastest reaching turnpike point k^*, x^* moving from the given initial point k^0, x^0. And using maximum principle [2-4], we will establish the structure of optimal control.
Proposition 2. Problems $(4)-(8)$ and $(19)-(23)$ have the same and unique solution.
Proof In [1], it proved that problems $(9)-(12)$ and $(14)-(16)$ are equivalent to corresponding time-optimal problems reaching turnpike points. Therefore, naturally the problems $(4)-(8)$ and $(19)-(23)$ must have the same solution as the problem $(4)-(8)$ is the direct union of above economical and ecological problems. The time- optimal problem is as follows

$$T \to \min, \tag{19}$$

$$\dot{x}(t) = ax(t)(1 - \frac{cx(t)}{\rho(1-s(t))Ak^{\alpha}(t)}) - u(t)x(t), \tag{20}$$

$$\dot{k}(t) = s(t)Ak^{\alpha}(t) - \mu k(t), \tag{21}$$

$$k(0) = k^0 > 0, k(T) = k^*, x(0) = x^0, x(T) = x^*, \tag{22}$$

$$0 \le u(t) \le \bar{u}, 0 \le s(t) \le 1-\varepsilon, 0 < W < a < \bar{u}. \tag{23}$$

There, we hardly change upper bound of admissible control s on ε, where ε sufficiently small and positive number.
Since the convexity of function $Y(K,L)$ and monotonicity of function

$$\frac{cx}{\rho(1-s)Ak^{\alpha}}$$

follows the uniqueness of solution. The proposition 2 is proved. According to maximum principle we write the Hamiltonian and conjugate system:

$$H(x, k, u, s, \psi_1, \psi_2) =$$

$$= \psi_2(ax(1 - \frac{cx}{\rho(1-s)Ak^\alpha}) - ux) + \psi_1(sAk^\alpha - \mu k) \tag{24}$$

$$\begin{cases} \dot{\psi}_1 = \psi_1(\mu - \alpha sAK^{\alpha-1}) - \psi_2 \cdot \frac{ac\alpha x^2}{\rho(1-s)AK^{\alpha+1}}, \\ \dot{\psi}_2 = -\psi_2 \cdot (a - \frac{2cax}{\rho(1-s)Ak^\alpha} - u). \end{cases} \tag{25}$$

From maximum principe(maximization on u), we have

$$u(t) = \begin{cases} 0 & , \quad \psi_2(t) > 0, \\ \bar{u} & , \quad \psi_2(t) < 0, \\ 0 \le u(t) \le \bar{u} & , \quad \psi_2(t) = 0. \end{cases} \tag{26}$$

But maximization on s reduces to the next result

$$\frac{\partial H}{\partial s} = 0 for 0 < s < 1, \frac{\partial H}{\partial s}/_{s=0} \le 0, \frac{\partial H}{\partial s}/_{s=1} \ge 0. \tag{27}$$

Since (27), we get

$$s(t) = \begin{cases} 0 & , \quad \psi_2 \ge f, \\ 1 - \varepsilon & , \quad \psi_2 \le \varepsilon^2 f), \\ 1 - \frac{x(t)}{Ak^\alpha(t)} \cdot \sqrt{\frac{ac\psi_2(t)}{\rho\psi_1}} & , \quad \varepsilon^2 f) < \psi_2 < f, \end{cases} \tag{28}$$

where $f(k, x, \psi_1) = \frac{A^2k^{2\alpha}\rho\psi_1}{acx^2}$.Note that formula (28) is correct when $\varphi_1\varphi_2 \ge 0$.

Conclusion

Assume that we consider optimal control problem with autonomous dynamic system and this problem has unique turnpike point. In this case, for long time period optimal process is situated at turnpike point for sufficiently long time period. Then original problem may be divided into two problems: time optimal problem and mathematical programming problem for finding the turnpike point. Namely, such method was used in the normalized mathematical model with finished products of biological origin. In future, need to construct computation method for the solution of this problem.

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