The method of successive approximations for exact solutions of Laplace equation and of heat-like and wave-like equations with variable coefficients

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Abstract

In this work, we present the method of successive approximations (shortly MSA) for obtaining the exact solution of Laplace equation with Dirichlet and Neumann boundary conditions and of heat-like and wave-like equations with variable coefficients. The results obtained by MSA are compared with known variational iteration method (VIM), homotopy-perturbation method (HPM), homotopy analysis method (HAM) and Adomian decomposition method (ADM) results. It is shown that all the above mentioned methods are equivalent for Laplace equation, heat-like and wave-like equations.

Keyword 1. *Method of Successive approximations; exact solution of Laplace equation; heat-like and wave-like equations.*

1 Introduction

Very recently, some promising exact analytical solution methods for Laplace equation are proposed, such as, He's variational iteration method (VIM) [1], homotopy-perturbation method (HPM)[3], Adomian decomposition method (ADM) [3,5] and homotopy analysis method (HAM)[2]. Some of these methods were also successfully employed to solve heat-like and wave-like equations with variable coefficients [4]. They give rapidly convergent successive approximations of the exact solution if such a solution exists. For concrete problems, a few number of approximations can be used for numerical purposes with high degree of accuracy. We aim in this work to effectively employ the well-known MSA to establish exact solutions for Laplace equation, heat-like and wave-like equations. In section 2 we present fixed-point iteration method. Based on this, we consider the MSA. In section 3 we apply the MSA to obtain the exact solutions for Laplace equation with Dirichlet and Neumann boundary conditions. We showed that the MSA is equivalent to VIM, HPM and ADM for Laplace equation. Moreover, we show that the MSA is also applicable for Helmholtz equation. In section 4 we apply the MSA to obtain the exact solutions for heat-like and wave-like equations with variable coefficients

2 Fixed-point iteration and method of successive approximations

Let

$$f(u) = 0 \tag{1}$$

be a nonlinear operator equation to be solved. The idea of fixed-point iteration consists of transforming Eq.(1) into an equivalent equation

$$u = \Phi(u) \tag{2}$$

and of constructing a sequence $\{u^n\}$ with the help of the iterative scheme

$$u^{n+1} = \Phi(u^n),\ n = 0, 1, \ldots \tag{3}$$

for a given starting value u^0. It is known that the iterative process (3) is convergent, if the operator Φ is contractive mapping with Lipschitz constant $\theta < 1$ [6]. The iteration (3) is usually called the method of successive approximations. As mentioned above, if the sequence $\{u^n\}$ is convergent, it gives the exact solution of Eq.(1) in the limit $n \to \infty$. In practice, however, if a sufficiently good initial approximation is known, only a few iterations are needed.

3 Exact solution of Laplace equation with Dirichlet and Neumann boundary conditions

We consider the Laplace equation

$$u_{xx} + u_{yy} = 0, \qquad 0 < x, y < \pi \tag{4}$$

subject to one of the following boundary conditions [1]

$$u(0, y) = 0 \ \ u(\pi, y) = \sin h\pi \cdot \cos y,$$

$$u(x, 0) = \sin hx, \ \ u(u, \pi) = -\sin hx, \tag{4a}$$

$$u(0, y) = \sin y,\ u(\pi, y) = \cos h\pi \cdot \sin y,$$

$$u(x, 0) = 0,\ u(x, \pi) = 0, \tag{4b}$$

$$u_x(0, y) = u_x(\pi, y) = 0,$$

$$u_y(x,y)=0,\ u_y(x,\pi)=2\cos 2x\cdot \sin h(2\pi) \tag{4c}$$

and

$$u_x(0,y)=u_x(\pi,y)=0,$$
$$u_y(x,0)=\cos x,\ u_y(x,\pi)=\cos hx\cdot \cos x. \tag{4d}$$

The exact solutions of (4), satisfying one of the boundary conditions (4a)-(4d) are $\sinh x\cdot\cos y$, $\cosh x\cdot\sin y$, $\cosh(2y)\cos 2x$ and $\sinh y\cdot\cos x$ respectively.

In order to use the MSA we must transform Eq.(4) into an equivalent one of kind (2). To this end we integrate Eq.(4) two times with respect to x on the interval $(0,x)$. As a result we have

$$u(x,y)=u(0,y)+xu_x(0,y)-\frac{\partial^2}{\partial y^2}\int_0^x d\xi\int_0^\xi u(x,y)dx. \tag{5}$$

Analogously, we can integrate Eq.(4) twice with respect to y on the interval $(0,y)$ and we get

$$u(x,y)=u(x,0)+yu_y(x,0)-\frac{\partial^2}{\partial x^2}\int_0^y d\eta\int_0^\eta u(x,y)dy. \tag{6}$$

The MSA (3) for (5) and (6) has a form

$$u^{n+1}(x,y)=u^n(0,y)+xu_x^n(0,y)-\frac{\partial^2}{\partial y^2}\int_0^x d\xi\int_0^\xi u^n(x,y)\qquad n=0,1,\dots \tag{7}$$

with $u^0(x,y)\neq 0$ and

$$u^{n+1}(x,y)=u^n(x,0)+yu_y^n(x,0)-\frac{\partial^2}{\partial x^2}\int_0^y d\eta\int_0^\eta u^n(x,y)dy\qquad n=0,1,\dots \tag{8}$$

with $u^0(x,y)\neq 0$, respectively. In (7) and (8) we have to choose the initial approximation $u^0(x,y)\neq 0$, satisfying at least two prescribed boundary conditions. We will use iteration (7) for Laplace equation with Dirichlet boundary conditions (4a) and (4b) and iteration (8) for Laplace equation with Neumann boundary conditions (4c) and (4d).

Proposition 1. *The MSA and VIM are equivalent for Laplace equation (4).*

Proof. The variational iteration method (VIM) for Laplace equation (4) gives the following approximations [1]

$$u^{n+1}(x,y) = u^n(x,y)+\int_0^x(\xi-x)\left(\frac{\partial^2u^n(\xi,y)}{\partial\xi^2}+\frac{\partial^2u^n(\xi,y)}{\partial y^2}\right)d\xi \tag{9a}$$

or

$$u^{n+1}(x,y) = u^n(x,y)+\int_0^y(\xi-y)\left(\frac{\partial^2u^n(x,\xi)}{\partial x^2}+\frac{\partial^2u^n(x,\xi)}{\partial\xi^2}\right)d\xi. \tag{9b}$$

Using integration by parts it is easy to show that

$$\int_0^x (\xi - x)\frac{\partial^2 u^n(\xi,y)}{\partial \xi^2}d\xi = xu_x^n(0,y) - u^n(x,y) + u^n(0,y) \tag{10}$$

and

$$\int_0^x (\xi - x)u^n(\xi,y)d\xi = -\int_0^x d\eta \int_0^\eta u^n(\xi,y)d\xi. \tag{11}$$

Substituting (10) and (11) into (9a) we get (7). Analogously proved the equivalence of (9b) and (8). □ □

Proposition 2. *The HPM and ADM are equivalent for Laplace equation, provided that* $\frac{\partial^2 u^0(x,y)}{\partial x^2} = 0$.

Proof. The homotopy-perturbation method (HPM) for Laplace equation (4) gives the approximation [3]:

$$u = \lim_{n\to\infty}(\nu^0 + \nu^1 + \nu^2 + \nu^3 + \cdots + \nu^n), \tag{12}$$

where

$$\begin{aligned} \frac{\partial^2 \nu^0}{\partial x^2} &= 0, \\ \frac{\partial^2 \nu^1}{\partial y^2} &= -\frac{\partial^2 \nu^0}{\partial y^2}, \\ \cdots & \quad \cdots \\ \frac{\partial^2 \nu^{n+1}}{\partial x^2} &= -\frac{\partial^2 \nu^n}{\partial y^2} \end{aligned} \tag{13}$$

while the Adomian decomposition method (ADM) for Laplace equation (4) gives the approximation [3]

$$u = \lim_{n\to\infty}(u^0 + u^1 + u^2 + \cdots + u^n), \tag{14}$$

where

$$u^{n+1}(x,y) = -L_{xx}^{-1}L_{yy}u^n(x,y),\ n = 0,1,\ldots\quad L_{xx} = \frac{\partial^2}{\partial x^2};\ L_{yy} = \frac{\partial^2}{\partial y^2}. \tag{15}$$

The comparison (13) and (15) shows that they are equivalent for Laplace equation when $\frac{\partial^2 u^0(x,y)}{\partial x^2} = 0$. □

Proposition 3. *The ADM and MSA are equivalent for Laplace equation provided that the initial approximation* $\bar{u}^0$ *of ADM is given by*

$$\bar{u}^0 = u^0(0,y) + xu_x^0(0,y) \tag{16}$$

or

$$\bar{u}^0(x,y) = u^0(x,0) + yu_y^0(x,0), \tag{17}$$

where u^0 *is an initial approximation of MSA.*

Proof. The ADM gives [3] approximations $\bar{u}^n(x,y)$ given by (15), i.e.

$$\bar{u}^{n+1} = -L_{xx}^{-1} L_{yy} \bar{u}^n(x,y),\ n = 0,1,\dots$$

Using this into (7) we have

$$\begin{aligned} u^1(x,y) &= \bar{u}^0 + \bar{u}^1, \\ u^2(x,y) &= \bar{u}^0 - L_{xx}^{-1} L_{yy} \bar{u}^0 - L_{xx}^{-1} L_{yy} \bar{u}^1 = \bar{u}^0 + \bar{u}^1 + \bar{u}^2, \\ \dots\dots\dots & \\ u^n(x,y) &= \bar{u}^0 + \bar{u}^1 + \bar{u}^2 \cdots + \bar{u}^n. \end{aligned}$$

It means that the n-th approximation u^n of MSA coincides with $\sum_{k=0}^{n} \bar{u_k}$, the partial sum of ADM. □ □

From the propositions 3.1, 3.2 and 3.3 immediately follows:

Theorem 2. *The MSA, VIM, HPM and ADM are equivalent for Laplace equation provided that* $\bar{u}^0 = u^0(0,y) + x \cdot u_x^0(0,y)$.

Proof. From proposition 3.1 and 3.3 it follows that $MSA \Leftrightarrow ADM$; $MSA \Leftrightarrow VIM$ hence $MSA \Leftrightarrow ADM \Leftrightarrow VIM$. From this and from proposition 3.2 it follows that all the above mentioned methods are equivalent. □ □

The MSA works well not only for Laplace equation, but also for Helmholtz equation

$$\Delta u + cu = 0. \tag{18}$$

It is easy to show that the equivalent equation to (18) is

$$u(x,y) = u(0,y) + x \cdot u_x(0,y) - \frac{\partial^2}{\partial y^2} \int_0^x d\xi \int_0^\xi u(\xi,y) d\xi - c \int_0^x d\xi \int_0^\xi u(\xi,y) d\xi.$$

For example, we consider equation (18) with $c = -2$ subject to boundary conditions:

$$\begin{aligned} u(0,y) &= e^y, \qquad u(1,y) = e^{1+y}, \\ u(x,0) &= e^x, \qquad u(x,1) = e^{1+x}. \end{aligned} \tag{19}$$

In this case the MSA is given by

$$u^{n+1}(x,y) = e^y + x \cdot u_x^n(0,y) - \frac{\partial^2}{\partial y^2} \int_0^x d\xi \int_0^\xi u^n(\xi,y) d\xi + 2 \int_0^x d\xi \int_0^\xi u^n(\xi,y) d\xi \quad n = 0,1,\dots \tag{20}$$

If we choose $u^0(x,y) = e^y(1+x)$, then from (20) we obtain the following successive approximations

$$\begin{aligned} u^1(x,y) &= e^y\left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}\right) \\ u^2(x,y) &= e^y\left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}\right) \end{aligned}$$

and

$$\lim_{n\to\infty} u^n(x,y) = e^{x+y},$$

which is the exact solution of (18), (19).

4 Exact solutions of heat-like and wave-like equations with variable coefficients

Recently, the VIM and Adomian method have been used to solve various kinds of heat-like and wave-like equations [4]. In this section the MSA is presented and is shown that all the above mentioned methods are equivalent for these equations.

4.1 We will use the MSA to obtain the exact solution of heat-like equation. We consider the initial-boundary value problems [4]

Example 1.

$$\begin{gathered} u_t = \frac{1}{2}x^2 u_{xx}, \qquad 0<x<1,\ t>0 \\ u(0,t)=0, \qquad u(1,t)=e^t, \\ u(x,0)=x^2. \end{gathered} \tag{21}$$

Example 2.

$$\begin{gathered} u_t = \frac{1}{2}(y^2u_{xx}+x^2u_{yy}), \qquad 0<x,y<1,\ t>0 \\ u_x(0,y,t)=0, \qquad u_x(1,y,t)=2\sinh t, \\ u_y(x,0,t)=0, \qquad u_y(x,1,t)=2\cosh t, \\ u(x,y,0)=y^2. \end{gathered} \tag{22}$$

Example 3.

$$\begin{gathered} u_t = x^4y^4z^4 + \frac{1}{36}\left(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}\right), \qquad 0<x,y,z<1,\ t>0 \\ u(0,y,z,t)=0,\ u(1,y,z,t)=y^4z^4(e^t-1), \\ u(x,0,z,t)=0,\ u(x,1,z,t)=x^4z^4(e^t-1), \\ u(x,y,0,t)=0,\ u(x,y,1,t)=x^4y^4(e^t-1), \\ u(x,y,z,0)=0. \end{gathered} \tag{23}$$

The exact solutions of Examples 1-3 are x^2e^t, $x^2\sinh t + y^2\cosh t$ and $(xyz)^4(e^t-1)$ respectively.
The VIM for (21) leads to the following iteration formula [4]

$$u^{n+1}(x,t) = u^n(x,t) - \int_0^t \left\{ \frac{\partial u^n(x,\xi)}{\partial \xi} - \frac{1}{2}x^2 \frac{\partial^2 u^n(x,\xi)}{\partial x^2} \right\} d\xi, n = 0,1,\dots \qquad (24)$$

while the MSA for (21) looks like

$$u^{n+1}(x,t) = u^n(x,0) + \frac{x^2}{2}\frac{\partial^2}{\partial x^2}\int_0^t u^n(x,\xi)d\xi, \quad n = 0,1,\dots \qquad (25)$$

If we take into account

$$\int_0^t \frac{\partial u^n(x,\xi)}{\partial \xi} d\xi = u^n(x,t) - u^n(x,0)$$

in (24), then we deduce that the VIM and MSA are equivalent methods for (21). Analogously, it is easy to show that the VIM and MSA are equivalent methods for (22) and (23).

4.2 Now we use the MSA to obtain the exact solution of wave-like equation. We consider next initial and boundary value problems [4]

Example 4.

$$\begin{gathered} u_{tt} = \frac{1}{2}x^2u_{xx}, \qquad 0 < x < 1,\ t > 0 \\ u(0,t) = 0,\ u(1,t) = 1 + \sinh t \\ u(x,0) = x,\ u_t(x,0) = x^2. \end{gathered} \qquad (26)$$

Example 5.

$$\begin{gathered} u_{tt} = \frac{1}{12}(x^2u_{xx} + y^2u_{yy}),\ 0 < x,y < 1, t > 0 \\ u_x(0,y,t) = 0,\ u_x(1,y,t) = 4\cosh t \\ u_y(x,0,t) = 0,\ u_y(x,1,t) = 4\sinh t \\ u(x,y,0) = x^4,\ u_t(x,y,0) = y^4. \end{gathered} \qquad (27)$$

Example 6.

$$
\begin{gathered}
u_{tt}=x^2+y^2+z^2+\frac{1}{2}(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}), \qquad 0<x,y,z<1,\ t>0\\
u(0,y,z,t)=y^2(e^t-1)+z^2(e^{-t}-1),\\
u(1,y,z,t)=(1+y^2)(e^t-1)+z^2(e^{-t}-1),\\
u(x,0,z,t)=x^2(e^t-1)+z^2(e^{-t}-1),\\
u(x,1,z,t)=(1+x^2)(e^t-1)+z^2(e^{-t}-1),\\
u(x,y,0,t)=(x^2+y^2)(e^t-1),\\
u(x,y,1,t)=(x^2+y^2)(e^t-1)+(e^{-t}-1),\\
u(x,y,z,0)=0, \qquad u_t(x,y,z,0)=x^2+y^2-z^2. \qquad (28)
\end{gathered}
$$

The exact solutions of Examples 4,5 and 6 are $x+x^2\sinh t$, $x^4\cosh t+y^4\sinh t$ and $(x^2+y^2)(e^t-1)+z^2(e^{-t}-1)$ respectively.
The VIM for an example 4 gives the following iteration formula [4]:

$$
u^{n+1}(x,t)=u^n(x,t)+\int_0^t(\xi-t)\left\{\frac{\partial^2u^n(x,\xi)}{\partial\xi^2}-\frac{1}{2}x^2\frac{\partial^2u^n(x,\xi)}{\partial x^2}d\xi,\right\} \qquad (29)
$$

while the MSA gives:

$$
u^{n+1}(x,t)=u^n(x,0)+u_t^n(x,0)t+\frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\int_0^t d\eta\int_0^\eta u^n(x,\xi)d\xi. \qquad (30)
$$

Substituting (10), (11) into (29), we get (30). It means that the VIM and MSA are equivalent for example 4. Analogously, it is easy to see that the VIM and MSA are equivalent methods for examples 5 and 6.

5 Comparison among MSA, HPM, VIM and ADM

In sections 3 and 4 we showed that these methods are equivalent for Laplace equations, heat-like and wave-like equations. For comparison, we present in Table 1 and 2 the selection of zeroth approximation $u^0(x,y)$ in various methods for Laplace equation and heat-like and wave-like equations respectively.

Table1 The initial approximation $u^0(x,y)$

Boundary condition	(4a)	(4b)	(4c)	(4d)
$VIM[1]$	$x\cos y$	$\left(1+\frac{x^2}{2}\right)\sin y$	$(1+2y^2)\cos 2x$	$\left(y+\frac{y^3}{3!}\right)\cos x$
$HAM[2]$	$x\cos y$	$\left(1+\frac{x^2}{2}\right)\sin y$	$(1+2y^2)\cos 2x$	$y\cos x$
$HPM[3]$	$x\cos y$	$1+\frac{x(\cosh\pi-1)}{\pi}$	$\cos 2x$	$y\cos x$
MSA	$x\cos y$	$\sin y$	$\cos 2x$	$y\cos x$
$ADM[3]$	$x\cos y$	$\sin y$	$\cos 2x$	$y\cos x$

Table2 The initial approximation $u^0(x,y)$

Examples	1	2	3	4	5	6
VIM[4]	x^2	y^2	$x^4y^4z^4t$	$x+x^2t$	x^4+y^4t	$(x^2+y^2-z^2)t$
MSA	x^2	y^2	0	x	x^4	0

All the above mentioned methods with initial approximations given in Table 1 and 2 give the exact solutions of considered problems.
It can be seen from the examples studied, that:

1. Comparison among MSA, HPM, VIM and ADM shows that although the results of these method when applied to the Laplace equation, heat-like and wave-like equations are the same, MSA does not require specific algorithms, such as ADM and VIM.
2. The MSA is much easier and more convenient than others considered above. The optimal identification of Lagrange multipliers via the variational theory can be difficult in VIM. In nonlinear problems arise the difficulties to calculate so-called Adomian polynomials, when using ADM [1].

6 Conclusion

In this paper, by the method of successive approximations , we obtain the exact solutions of Laplace equation and various kinds of heat-like and wave-like equations. MSA, like HPM, does not require specific algorithms and complex calculations such as ADM or construction of correction functionals using general Lagrange's multipliers in VIM. It may be concluded that this method is very powerful and efficient one in finding exact solutions for wide classes of problems.

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