Remarks on the Ritt condition

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Abstract

We study the Ritt condition for a quasinilpotent operator pencils on the real line. Also obtained some estimation of consecutive differences for concrete family.

1 Introduction and Preliminaries

Throughout this T will denote a bounded linear operator on a Banach space X.

An operator T is called power bounded, if

$$\sup_{n\geq 0} ||T^n|| < +\infty.$$

Recall that for an operator T the spectrum

$$\sigma(T) = \{\lambda \in \mathbb{C} : \text{ such that } \lambda I - T \text{ is not invertible}\}$$

is always a compact subset of the complex plane. Thus the resolvent set $\rho(T) = \mathbb{C}\backslash\sigma(T)$ is an open set. The operator valued analytic function $\lambda \in \rho(T) \to R(\lambda, T) = (\lambda I - T)^{-1}$ is a called the resolvent of T.

Recall that the Ritt condition of a bounded operator T on a Banach space is

$$||R(\lambda, T)|| \leq \frac{\text{const}}{|\lambda - 1|}, \quad |\lambda| > 1,$$

which is equivalent to a geometric condition much stronger than power boundedness of T, namely,

$$\sup_{n\geq 0} n||T^n - T^{n+1}|| < +\infty$$

has to be added to the power boundedness of T, see [6].

Namely, $T : X \to X$ is a Ritt operator, if there exits two constants $C_0, C_1 > 0$ such that

$$\forall n \geq 0, \quad ||T^n|| \leq C_0,$$
$$\forall n \geq 1, \quad n||T^n - T^{n+1}|| \leq C_1.$$

This is equivalent to the Ritt condition is

$$\sigma(T) \subset \overline{\mathbb{D}}$$

and there exits $C > 0$ such that

$$\forall \lambda \notin \overline{\mathbb{D}}, \quad ||(\lambda I - T)^{-1}|| \leq \frac{\text{const}}{|\lambda - 1|}.$$

This implies that $\exists \gamma \in (0, \frac{\pi}{2}) : \sigma(T) \subset \overline{B_\gamma}$, where B_γ is the interior of the convex hull of 1 and of the disc of radius $\sin\gamma$ centered at 0. $(0 < \gamma < \frac{\pi}{2})$

Denote by V the classical Volterra operator

$$(Vf)(x) = \int_0^x f(s)ds, \quad 0 < x < 1, \quad \text{on} \quad L^p(0,1), \quad 1 \leq p \leq \infty.$$

It is well known and easy to see that V acts boundedly on $L^p(0,1)$. $(1 \leq p \leq \infty)$ The Volterra operator is quasinilpotent, $(\sigma(V) = \{0\})$ but not nilpotent. Hence e^{-tV} is a semigroup of all contraction in $L^2(0,1)$. The identity operator will be denote by I. Pedersen proved that $I - V = S^{-1}(I + V)^{-1}S$, where $(Sf)(x) = e^{-x}f(x)$. (see [1]) From the latter similarity and the fact that $(I + V)^{-1}$ acting on $L^2(0,1)$ has norm 1, (see Halmos' Problem Book ([3], Problem 150)) it immediately follows that $I - V$ acting on $L^2(0,1)$ is power bounded. In [5], [8], [9] studied power boundedness of Laguerre polynomials and an operator norm of $\varphi(V) = \sum_{k=0}^n a_k V^k$ $(a_n \neq 0, n \geq 1)$ on $L^2(0,1)$.

2 The Results

Proposition 1. ([9]) Let $\sigma(Q) = \{0\}$. If $I - Q$ satisfies the Ritt condition, then so does $I - tQ$ for all $t \geq 0$.

Proposition 2. Let $\sigma(Q) = \{0\}$. The operators $I - Q$ and $I + Q$ are power bounded if and only if $Q = 0$.

Proof. We can write

$$Q = Q\left(\frac{I - Q + I + Q}{2}\right)^n = \frac{1}{2^n}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(I - Q)^{n-k}Q(I + Q)^k.$$

Observe that, for large n, either $||(I-Q)^{n-k}Q||$ or $||Q(I+Q)^k||$ is small, by ([2], Theorem 9.1), while the remaining operator powers (actually both $||(I+Q)^k||$ and $||(I-Q)^{n-k}||$) are bounded by assumption. It follows that $Q=0$. Conversely, it will clear. □

Proposition 3. Let A and B be two commuting Ritt operators. Then their product AB is also a Ritt operator. Consequently, $(1-t)I+t(I-Q)^2$ is a Ritt operator for all $0 \le t \le 1$.

Theorem 1. Let $\sigma(Q)=\{0\}$. If $I-Q$ is a Ritt operator then $I-tQ$ is Ritt operator for all $t>0$. Conversely, $I-tQ$ is Ritt operator for some $t>0$, $(t \neq 1)$ then $I-Q$ is a Ritt operator.

Proof. Suppose that $I-Q$ is a Ritt operator then so does $I-tQ$ is a Ritt for all $t>0$ by Proposition 1. If $I-tQ$ is a Ritt operator for some $t>0$, $(t \neq 1)$ then also $I-stQ$ is a Ritt operator for all $s>0$ by Proposition 1. Take here $s=\frac{1}{t}$, so $I-Q$ is a Ritt operator. For $t<0$, this implication is not true, by Proposition 2. □

Corollary 1. If $\varphi(V)$ is power bounded, then so is $\phi_\alpha(V)=(1-s)I+s\varphi(V)$ for all $0 \le s \le 1$.

This statement can be used to immediately derive the estimate

$$||\varphi(V)^{n+1}-\varphi(V)^n||_2 = O\left(\frac{1}{\sqrt{n}}\right)$$

from [7], Theorem 4.5.3. In particular, we take

$$\varphi(V)=L_m(V)=\sum_{k=0}^{m}\left(\begin{array}{c} m \\ k \end{array}\right)(-1)^k\frac{V^k}{k!}, \quad m \ge 1,$$

(m-th Laguerre polynomials) we obtain

$$||L_m(V)^{n+1}-L_m(V)^n||_2 = O\left(\frac{1}{\sqrt{n}}\right).$$

If $\varphi(V)=e^{-tV}$ for all $t>0$, then we get

$$||e^{-t(n+1)V}-e^{-tnV}||_2 = O\left(\frac{1}{\sqrt{n}}\right).$$

Remark 1. The operator $I-V$ is power bounded, but not Ritt operator. (As observed by Urszula Skórnik, 1999)

In contrast $I-J^\alpha$, where

$$(J^\alpha f)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt$$

(the Riemann-Liouville intergarl operator) is a Ritt operator in $L^p(0,1)$ $(1 \leq p \leq \infty)$, if $0 < \alpha < 1$ (see [4]). Where Γ is the Euler gamma function. In particular $V = J^1$.

Remark 2. Theorem 1 yields the corresponding information about the power boundedness of the Sarason operator pencil.

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