A finite covolume discretization for the Navier-Stokes equations

D. Bayanjargal[1] and DO Y. Kwak[2]
[1]*School of Economic Studies, NUM, Mongolia,*
[2]*Mathematical Science Department, KAIST, Korea*
e-mail: bayanjargal@ses.edu.mn, kdy@kaist.ac.kr

Abstract

We propose a new discretization method for the Navier-Stokes equations which is a variant of covolume scheme. There are two ways to introduce the covolume approximation to the Navier-Stokes equations. One uses the divergence (or conservative) form of Navier-Stokes equations which we call the conservative covolume method, the another uses its original form. In this paper, we present the latter one. Primal and dual grids are used in the covolume method. The finite element space for the velocity is the Crouzeix-Raviart space for triangles consisting of piecewise linear functions and the finite element space for the pressure is the space of piecewise constant functions on the primal elements, whereas the test function space for the velocity consists of certain piecewise constant functions on the dual elements. The general theory based on the results of approximation for branches of nonsingular solutions of nonlinear problems gives us an opportunity to study of the convergence of the covolume method for the Navier-Stokes equations. Efficiency of the proposed method has been tested on a number of test problems.

1 Introduction

The finite element discretizations for the Navier-Stokes problem are based on its variational formulation in appropriate function spaces, and approximate it in certain finite dimensional subspaces consisting of piecewise polynomial functions. The general theory for the finite element and mixed finite element methods for the Navier-Stokes equations and Stokes problem has been introduced in [1], [2], [3], [4], [9], [11], [12], [13], [15].

The purpose of this paper is to introduce a nonconforming covolume method for the Navier-Stokes equations on triangular-quadrilateral grids and prove its convergence. In [6], Chou first introduced a covolume method for the Stokes problem. Chou and Kwak proposed a MAC-like covolume method for the Stokes problem in [7]. In their works, the trial functions for the velocity are piecewise linear functions and for the pressure are piecewise constant functions on the primal elements. The test functions are piecewise constant on the dual elements. Similar nonconforming space for rectangular grid is introduced in [14] for which a parallel covolume method can be described as in [8].

The difficulty of the Navier-Stokes problem comes from its nonlinear (convection) term. We analyze the covolume approximation using the approximation of branches of nonsingular solutions which is presented in [11]. The analysis, based on a general form of the implicit function theorem, is a variant of a broader theory developed by Brezzi, Rappaz and Raviart [5].

The discrete system resulting from the covolume scheme for the Navier-Stokes equations constitutes the nonlinear system of algebraic equations. For solving the nonlinear system, we use Picard iteration method and Uzawa algorithm with conjugate gradient for solving the linear system.

2 Homogeneous Navier-Stokes equations

Let Ω be a Lipschitz, bounded domain in $\mathbb{R}^2$ with boundary Γ. Let $\mathbf{f} \in L^2(\Omega)$ be a given vector function. We are looking for a vector function $\mathbf{u} = (u_1, u_2)$ and a scalar function p, representing the velocity and the pressure of the fluid, which are defined in Ω and satisfy the following equations and boundary conditions:

$$-\nu\,\Delta\mathbf{u} + \sum_{i=1}^{2} u_i D_i \mathbf{u} + \mathbf{grad} p = \mathbf{f} \qquad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{2.2}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma, \tag{2.3}$$

where ν is the coefficient of viscosity, a constant.

The stationary linearized form of the Navier-Stokes equations is the stationary Stokes equations:

$$-\nu\,\Delta\mathbf{u} + \mathbf{grad}\, p = \mathbf{f} \qquad \text{in } \Omega, \tag{2.4}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{2.5}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma. \tag{2.6}$$

The finite element discretization of the Navier-Stokes equations is based on its variational formulation. To do this, we use the Sobolev spaces. They are based on

$$L^2(\Omega) = \big\{ v : \int_\Omega |v|^2 dx = \|v\|_0^2 < +\infty \big\},$$

the space of square integrable functions on Ω. We then define the Sobolev spaces, for k nonnegative integer,

$$H^k(\Omega) = \big\{ u \in L^2(\Omega) : \ D^\alpha u \in L^2(\Omega), \quad \forall |\alpha| \le k \big\},$$

where $D^\alpha u$ denotes any and all derivatives of order k:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index, α_i, $1 \leq i \leq n$, are nonnegative integers and

$$|\alpha| = \sum_{i=1}^{n} \alpha_i,$$

these derivatives being taken in the sense of distributions. The Sobolev inner product is

$$(u, v)_k = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v), \tag{2.7}$$

summation being taken over all α such that $|\alpha| \leq k$. The corresponding Sobolev norm is

$$\|u\|_k = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_0^2 \right)^{1/2}, \tag{2.8}$$

and semi-norm

$$|u|_k = \left(\sum_{|\alpha| = k} \|D^\alpha u\|_0^2 \right)^{1/2}. \tag{2.9}$$

It is clear that $H^0(\Omega) = L^2(\Omega)$. Our particular interest is the space

$$L_0^2(\Omega) = \left\{ u \in L^2(\Omega);\ \int_\Omega u(x) dx = 0 \right\}$$

consists of square integrable functions having zero mean over Ω, the space $H^1(\Omega)$, and the subspace

$$H_0^1(\Omega) = \{u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma\}$$

whose elements vanish on the boundary Γ.

We shall work with 2 dimensional vector functions with components in one of these spaces. We use the notation

$$\mathbf{H}^1(\Omega) = (H^1(\Omega))^2, \quad \mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^2.$$

We now define the bilinear forms

$$a_0(\mathbf{u}, \mathbf{v}) = \nu \int_\Omega \mathbf{grad}\ \mathbf{u} : \mathbf{grad}\ \mathbf{v}\, dx \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.10}$$

$$b(\mathbf{v}, q) = -\int_\Omega q \operatorname{div} \mathbf{v}\, dx \quad \text{for all } \mathbf{v} \in \mathbf{H}^1(\Omega),\ q \in L^2(\Omega), \tag{2.11}$$

and the trilinear form

$$c(\mathbf{w}; \mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^{2} \int_\Omega w_i (D_i u_j) v_j\, dx \quad \text{for all } \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega). \tag{2.12}$$

Here

$$\mathbf{grad}\ \mathbf{u} : \mathbf{grad}\ \mathbf{v} := \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}.$$

Then, the variational formulation of (2.1)-(2.3), reads as follows: Find functions $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} a(\mathbf{u};\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ b(\mathbf{u},q) = 0 \quad \text{for all } q \in L_0^2(\Omega), \end{aligned} \tag{2.13}$$

where

$$a(\mathbf{u};\mathbf{u},\mathbf{v}) = a_0(\mathbf{u},\mathbf{v}) + c(\mathbf{u};\mathbf{u},\mathbf{v}).$$

Now, we can define the following space

$$V = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega);\ b(\mathbf{v},q) = 0 \quad \text{for all } q \in L_0^2(\Omega)\},$$

which consists of (weakly) divergence free functions. Then Problem (2.13) has the equivalent form: Find a pair $(\mathbf{u},p) \in V \times L_0^2(\Omega)$ such that:

$$a(\mathbf{u};\mathbf{u},\mathbf{v}) - (p, \text{div}\ \mathbf{v}) = (\mathbf{f}, v) \quad \forall \mathbf{v} \in H_0^1(\Omega). \tag{2.14}$$

The following theorems show the existence and uniqueness of solutions (2.14) which are proved in [11].

Theorem 2.1: [11] Let $N \leq 3$ and let Ω be a bounded domain of R^N with a Lipschitz-continuous boundary Γ. Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, there exists at least one pair $(\mathbf{u},p) \in V \times L_0^2(\Omega)$ solution of (2.14) or equivalently solution of (2.1)-(2.3).

Theorem 2.2: [11] There exists a unique solution $(\mathbf{u},p)$ in $V \times L_0^2(\Omega)$ of (2.14) if the viscosity ν and the data $\mathbf{f}$ satisfy

$$(\mathcal{N}/\nu^2)\|\mathbf{f}\|_{\mathbf{H}^{-1}} < 1, \tag{2.15}$$

where $\mathbf{H}^{-1}(\Omega)$ is the dual space of $\mathbf{H}^1(\Omega)$,

$$\|\mathbf{f}\|_{\mathbf{H}^{-1}} = \sup_{\mathbf{v}\in\mathbf{H}_0^1} \frac{(\mathbf{f},\mathbf{v})}{|\mathbf{v}|_1}. \tag{2.16}$$

and for the trilinear form $c(\mathbf{w};\mathbf{u},\mathbf{v})$,

$$\mathcal{N} = \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{H}_0^1} \frac{c(\mathbf{w};\mathbf{u},\mathbf{v})}{|\mathbf{u}|_1|\mathbf{v}|_1|\mathbf{w}|_1}. \tag{2.17}$$

3 Covolume discretization

To discretize the Navier-Stokes equation using the covolume method, we need to define the primal and dual grids. First, we define a primal grid. Let T_h be a primal decomposition of the domain $\bar{\Omega}$ into a union of triangular elements K_B such that the usual regularity conditions are satisfied:

- $\bar{\Omega} = \bigcup\{K_B \in T_h\}$ where K_B is triangle whose barycenter is B.
- Any two triangles K_{B_1}, K_{B_2} only intersect in common faces, edges or vertices.
- $\mathrm{diam}(K_B) \leq h$ for each $K_B \in T_h$.

The nodes of an element are the midpoints of its sides. Let N be the number of nodal points and N_T be the number of triangles in the decomposition T_h. We denote by $P_1, P_2, \ldots, P_{N_S}$ those nodes belonging to the interior of Ω and $P_{N_{S+1}}, ..., P_N$ those on the boundary.

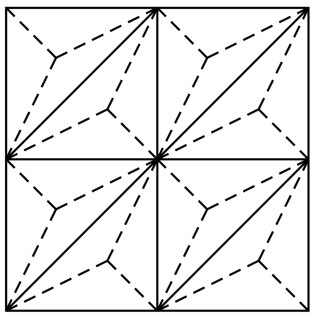

Figure.1 Primal and dual grids

The finite element space ${\mathbf{H}_0}^h$ for the velocity is the Crouzeix-Raviart space for triangles or nonconforming P_1 element [9]:

$$\begin{aligned}\mathbf{H}_0^h = \{\mathbf{v}_h \in (L^2(\Omega))^2 : \mathbf{v}_h|_{K_B} \in (P_1(K_B))^2,\ \ \forall K_B \in T_h,\\ \mathbf{v}_h \text{ are continuous at the midpoints of the triangle edges and}\\ \mathbf{v}_h = 0 \text{ at the midpoints of the edges on } \partial\Omega\},\end{aligned}$$

where $P_1(K_B)$ denotes the piecewise linear function on the triangle K_B and the finite space L_0^h for the pressure is:

$$L_0^h = \{q_h \in L_0^2(\Omega) : q_h|_{K_B} \text{ is constant},\ \ \forall K_B \in T_h\}.$$

Since $\mathbf{H}_0^h$ is nonconforming, the gradient and divergence operator on it must be defined piecewise:

$$(\nabla_h \mathbf{v}_h)|_{K_B} := \nabla(\mathbf{v}_h|_{K_B}),$$
$$(\text{div}_h\ \mathbf{v}_h)|_{K_B} := \text{div}\ (\mathbf{v}_h|_{K_B}).$$

On the space $\mathbf{H}_0^h$ we define the mesh dependent norms:

$$\|\mathbf{v}\|_{1,h}^2 = \sum_{i=0}^{1} |\mathbf{v}_h|_{i,K_B}^2 \quad \text{and} \quad |\mathbf{v}_h|_{i,K_B}^2 = \sum_{K_B \in T_h} \int_{K_B} |\partial_i \mathbf{v}_h|^2$$

which are also called broken norms. Below we shall use ∇ for ∇_h and div for div_h for our convenience when there is no confusion.

Next we construct a dual grid by connecting the barycenters of the primal elements. Let the dual decomposition be $T_h^* = \bigcup K_p^*$. The dual grid is a union of interior quadrilaterals and border triangles.

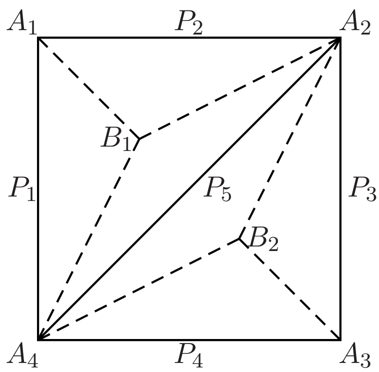

Figure.2

For example, referring to Fig.2, the interior node P_5 belongs to the common side of the triangles $K_{B_1} = \triangle A_1A_2A_4$ and $K_{B_2} = \triangle A_2A_3A_4$ and the quadrilaterals $B_1A_2B_2A_4$ is the dual element with node at P_5. For a boundary node like P_3 the associated dual element is a triangle $\triangle A_2A_3B_2$.

We shall denote the test function space associate with the dual decomposition by $\mathbf{Y}_h$, the space of certain piecewise constant vector functions. That is

$$\mathbf{Y}_0^h = \{q \in (L^2(\Omega))^2 : q|_{K_p^*} \text{ is a constant vector, and}$$
$$q|_{K_p^*} = 0 \text{ on any boundary dual element } K_p^*\}.$$

Denote by χ_j^* the scalar characteristic function associated with the dual element $K_{P_j}^*, j = 1, 2, ..., N_S$. We see that for any $\mathbf{v}_h \in \mathbf{Y}_0^h$

$$\mathbf{v}_h(x) = \sum_{j=1}^{N_s} \mathbf{v}_h(P_j)\chi_j^*(x) \qquad \forall x \in \Omega. \tag{3.1}$$

We now describe a covolume approximation for the stationary Navier-Stokes equations. As for the approximate pressure space L_0^h, we define it to be the set of all piecewise constants with respect to the primal partition since in the covolume method the pressure is assigned at the centers of triangular elements. The test and trial function spaces should reflect the fact that in the covolume method the momentum equation (2.1) is integrated over the dual element and the continuity equation (2.2) over the primal element.

For $\mathbf{u}_h \in \mathbf{H}_0^h, \mathbf{v}_h \in \mathbf{Y}_0^h, p_h, q_h \in L_0^h$ and $\mathbf{f} \in \mathbf{H}^{-1}$, define the following trilinear form:

$$\begin{aligned} c^*(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) :&= \sum_{i,j=1}^{2} \int_\Omega u_{hi}(D_i u_{hj}) v_{hj}\, dx \\ &= \sum_{i,j=1}^{2} \sum_{k=1}^{N_S} v_{hj}(P_k) \int_{K^*_{P_k}} u_{hi}(D_i u_{hj})\, ds, \end{aligned} \tag{3.2}$$

bilinear forms:

$$a_0^*(\mathbf{u}_h,\mathbf{v}_h) := -\nu \sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \cdot \int_{\partial K^*_{P_i}} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}}\, ds, \tag{3.3}$$

$$b^*(\mathbf{v}_h,p_h) := \sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \int_{\partial K^*_{P_i}} p_h \mathbf{n}\, ds \tag{3.4}$$

$$d^*(\mathbf{u}_h,q_h) := -\sum_{k=1}^{N_T} q_h(B_k) \int_{K_{B_k}} div\ \mathbf{u}_h dx, \tag{3.5}$$

and

$$(\mathbf{f}, v_h) := \sum_{i=1}^{N_S} \mathbf{v}_h(P_i) \int_{K^*_{P_i}} \mathbf{f}\ dx. \tag{3.6}$$

Equation (3.2) and (3.3) are obtained by integrating the second and first terms of (2.1) against test functions, respectively and then using the second Green's identity.

Then the approximate formulation for (2.1)-(2.3) is: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_0^h \times L_0^h$ such that

$$\begin{aligned} a^*(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) + b^*(\mathbf{v}_h,p_h) &= (\mathbf{f},\mathbf{v}_h),\ \forall \mathbf{v}_h \in \mathbf{Y}_0^h \\ d^*(\mathbf{u}_h,q_h) &= 0,\quad \forall q_h \in L_0^h \end{aligned} \tag{3.7}$$

where

$$a^*(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) = a_0^*(\mathbf{u}_h,\mathbf{v}_h) + c^*(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h).$$

Define one to one transfer operator γ_h from $\mathbf{H}_0^h$ onto $\mathbf{Y}_0^h$ by

$$\gamma_h \mathbf{v}_h = \sum_{j=1}^{N_S} \mathbf{v}_h(P_j)\chi_j^*(x) \qquad \forall x \in \Omega. \tag{3.8}$$

for all $\mathbf{v}_h \in \mathbf{H}_0^h$. Using the transfer operator γ_h, we redefine the bilinear and trilinear forms in (3.7) as follows. For all $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{H}_0^h$ and $q_h \in L_0^h$

$$A_0(\mathbf{u}_h, \mathbf{v}_h) := a_0^*(\mathbf{u}_h, \gamma_h \mathbf{v}_h) \tag{3.9}$$

$$C(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) := c^*(\mathbf{u}_h; \mathbf{u}_h, \gamma_h \mathbf{v}_h) \tag{3.10}$$

$$B(\mathbf{v}_h, q_h) := b^*(\gamma_h \mathbf{v}_h, q_h), \tag{3.11}$$

and

$$A(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) := A_0(\mathbf{u}_h, \mathbf{v}_h) + C(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h). \tag{3.12}$$

It is shown in [6] that the bilinear form A_0 is symmetric and that the two bilinear forms B and d^* are identical. Hence the approximation problem (3.7) becomes: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_0^h \times L_0^h$ such that

$$\begin{aligned} A(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + B(\mathbf{v}_h, p_h) &= (\mathbf{f}, \gamma_h \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_0^h \\ B(\mathbf{u}_h, q_h) &= 0, \quad \forall q_h \in L_0^h. \end{aligned} \tag{3.13}$$

Since the redefined forms are defined only on the spaces $\mathbf{H}_0^h$ and L_0^h, reformulations help one to analyze the scheme using finite element techniques.

In order that (3.13) is a stable approximation to (2.13), as $h \to 0$, it is crucial that the spaces $\mathbf{H}_0^h \times L_0^h$ satisfy a compatibility condition, which is called "inf-sup" or "Babuska-Brezzi" condition,

$$\inf_{q_h \in L_0^h} \Big(\sup_{\mathbf{v}_h \in \mathbf{H}_0^h} \frac{b_h(q_h, \mathbf{v}_h)}{\|q_h\| \|\mathbf{v}_h\|_{1,h}} \Big) \geq \beta > 0, \tag{3.14}$$

where the constant β is reguired to be independent of h. Chou [6] proved the following theorem for the covolume approximation for Stokes equations:

$$\begin{aligned} A_0(\mathbf{u}_h, \mathbf{v}_h) + B(\mathbf{v}_h, p_h) &= (\mathbf{f}, \gamma_h \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_0^h \\ B(\mathbf{u}_h, q_h) &= 0, \quad \forall q_h \in L_0^h. \end{aligned} \tag{3.15}$$

Theorem 3.1: [6] Let the triangulation family of the domain Ω be quasi-uniform, let $(\mathbf{u}_h, p_h)$ be the solution of the problem (3.15), and $(\mathbf{u}, p)$ solve the problem (2.4)-(2.6). Then there exists a positive constant C independent of h such that

$$|\mathbf{u} - \mathbf{u}_h|_{1,h} + \|p - p_h\|_0 \leq Ch(\|\mathbf{u}\|_2 + \|p\|_1 + 1), \tag{3.16}$$

provided that $\mathbf{u} \in H_0^1(\Omega) \bigcap \mathbf{H}^2(\Omega), p \in H^1(\Omega)$. Furthermore,

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq Ch(\|\mathbf{u}\|_2 + \|p\|_1 + 1). \tag{3.17}$$

4 Convergence of the covolume method

Before we study the convergence of the covolume method, we begin with a brief introduction to the study of the numerical approximation of solutions of nonlinear problems.

Let X and Y be two Banach spaces and Λ a compact interval of the real line $\mathbb{R}$. We set the following class of problems:

$$F(\lambda, u) = u + TG(\lambda, u), \tag{4.1}$$

where $T \in \mathcal{L}(Y; X)$, G is $\mathcal{C}^2$ mapping from $\Lambda \times X$ into Y.
We want to find pairs $(\lambda, u) \in \Lambda \times X$ solutions of

$$F(\lambda, u) = 0. \tag{4.2}$$

where $F : \Lambda \times X \to X$. We shall assume that there exists a compact interval $\Lambda \subset R$ and a branch $\{(\lambda, u(\lambda)); \lambda \in \Lambda\}$ of nonsingular solutions of (4.2) which means that $\lambda \to u(\lambda)$ is continuous function from Λ into X and

$$F(\lambda, u(\lambda)) = 0. \tag{4.3}$$

Moreover, we assume that these solutions are nonsingular in the sense that:

$$D_u F(\lambda, u(\lambda)) \text{ is an isomorphism of } X \text{ for all } \lambda \in \Lambda. \tag{4.4}$$

Now we study the approximation of the branch of nonsingular solutions. For each value of real parameter $h > 0$ which will tend to zero, we are given a finite dimensional subspace X_h of the space X and an operator $T_h \in \mathcal{L}(Y; X_h)$ intended to approximate T. We set:

$$F_h(\lambda, u_h) = u_h + T_h G(\lambda, u_h), \qquad \lambda \in \lambda, u_h \in X_h. \tag{4.5}$$

Then, the approximate problem consists in finding pairs $(\lambda, u_h) \in \Lambda \times X_h$, solutions of

$$F_h(\lambda, u_h) = 0. \tag{4.6}$$

The following theorem shows the sufficient conditions ensuring the existence and uniqueness of a branch $(\lambda, u_h(\lambda)) \in \Lambda \times X_h$ of solutions of (4.6) in a suitable neighborhood of the branch solutions of (4.3).

Theorem 4.1: [11] Assume that G is a C^2 mapping from $\Lambda \times X$ into Y and the mapping $D^2 G$ is bounded on all bounded subsets of $\Lambda \times X$. Assume in addition that the following conditions hold:

(i). There exists another Banach space Z contained in Y, with continuous imbedding, such that

$$D_u G(\lambda, u) \in \mathcal{L}(X; Z) \quad \forall \lambda \in \Lambda, \quad \forall u \in X. \tag{4.7}$$

(ii). We assume that

$$\lim_{h \to 0} \|(T_h - T)g\|_X = 0 \quad \forall g \in Y \tag{4.8}$$

and

$$\lim_{h \to 0} \|T_h - T\|_{\mathcal{L}(Z;X)} = 0. \tag{4.9}$$

Let $(\lambda, u(\lambda)); \lambda \in \Lambda$ be a branch of nonsingular solutions of (4.3). Then there exists a neighborhood $\mathcal{O}$ of the origin in X and for $h \leq h_0$ small enough a unique C^2 function $\lambda \in \Lambda \to u_h(\lambda) \in X$ such that:

$$(\lambda, u_h(\lambda)); \lambda \in \Lambda \text{ is a branch of nonsingular solutions of (4.6)}, \tag{4.10}$$

$$u_h(\lambda) - u(\lambda) \in \mathcal{O} \quad \forall \lambda \in \Lambda. \tag{4.11}$$

Furthermore, there exists a constant $K > 0$ independent of h and λ with:

$$\|u_h(\lambda) - u(\lambda)\|_X \leq K \|(T_h - T)G(\lambda, u(\lambda))\|_X \quad \forall \lambda \in \Lambda. \tag{4.12}$$

Let us define

$$X_h := \mathbf{H}_0^h \times L_0^h,$$

and a Banach space $\tilde{X}$ as:

$$\tilde{X} := X \oplus X_h. \tag{4.13}$$

Although Theorem 4.1 is originally stated for the Navier-Stokes equations when

$$X = \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega), \quad Y = \mathbf{H}^{-1}(\Omega) \tag{4.14}$$

and

$$\mathbf{H}_0^h \subset \mathbf{H}_0^1(\Omega), \quad L_0^h \subset L_0^2(\Omega),$$

it still holds when X is replaced by $\tilde{X}$ and the norm $\|\cdot\|_X$ by the broken norm which is defined in Section 3.

Thus, it gives us an opportunity to apply the results of Theorem 4.1 to prove our main theorem.

Now we recall the approximation problem (3.13) for (2.13):

Find a pair $(\mathbf{u}_h, p_h) \in \mathbf{H}_0^h \times L_0^h$ solution of

$$\begin{aligned} A(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) &= (\mathbf{f}, \gamma \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_0^h, \\ (q_h, \ \operatorname{div} \mathbf{u}_h) &= 0 \quad \forall q_h \in L_0^h. \end{aligned} \tag{4.15}$$

In order to study (4.15) we need the following hypotheses:
Hypothesis H1
(Approximation property of $\mathbf{H}_0^h$). *There exists an operator* $r_h \in \mathcal{L}([H^2(\Omega) \bigcap H_0^1(\Omega)]^2; \mathbf{H}_0^h)$ *such that:*

$$\|\mathbf{v} - r_h\mathbf{v}\|_1 \leq Ch\|\mathbf{v}\|_2 \quad \forall \mathbf{v} \in \mathbf{H}^2(\Omega). \tag{4.16}$$

Hypothesis H2
(Approximation property of L_0^h). *There exists an operator* $S_h \in \mathcal{L}(L^2(\Omega); L_0^h)$ *such that:*

$$\|q - S_hq\|_0 \leq Ch\|q\|_1 \quad \forall q \in H^1(\Omega). \tag{4.17}$$

Hypothesis H3
(Uniform inf-sup condition). *For each* $q_h \in L_0^h$ *there exists a* $\mathbf{v}_h \in \mathbf{H}_0^h$ *such that:*

$$(q_h, \mathrm{div}\mathbf{v}_\mathrm{h}) = \|\mathrm{q_h}\|_0^2, \quad |\mathbf{v}_\mathrm{h}|_1 \leq \mathrm{C}\|\mathrm{q_h}\|_0, \tag{4.18}$$

with a constant $C > 0$ *independent of* h, q_h *and* $\mathbf{v}_h$.

Then we have the following result:

Theorem 4.2: Assume that the hypotheses $\mathbf{H1}, \mathbf{H2}$ and $\mathbf{H3}$ hold. Let

$$\{(\lambda, \mathbf{u}(\lambda), \lambda p(\lambda)); \lambda = 1/\nu \in \Lambda\}$$

be a branch of nonsingular solutions of the Navier-Stokes (2.13). Then there exists a neighborhood $\mathcal{O}$ of the origin in $\tilde{X}$ and for $h \leq h_0$ sufficiently small a unique C^∞ branch $\{(\lambda, \mathbf{u}_h(\lambda), \lambda p_h(\lambda)); \lambda = 1/\nu \in \Lambda\}$ of nonsingular solutions of (4.15) such that:

$$(\mathbf{u}_h(\lambda), \lambda p_h(\lambda)) \in (\mathbf{u}(\lambda), \lambda p(\lambda)) + \mathcal{O} \quad \forall \lambda \in \Lambda.$$

Moreover, we have the convergence property:

$$\lim_{h\to 0} \sup_{\lambda \in \Lambda} \{|\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda)|_1 + \|p_h(\lambda) - p(\lambda)\|_0\} = 0. \tag{4.19}$$

In addition, if the mapping $\lambda \to (\mathbf{u}(\lambda), p(\lambda))$ is continuous from Λ into $\mathbf{H}^2(\Omega) \times H^1(\Omega)$, then

$$|\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda)|_1 + \|p_h(\lambda) - p(\lambda)\|_0 \leq Kh. \tag{4.20}$$

is satisfied for all $\lambda \in \Lambda$.

Proof. We shall use the results of Theorem 4.1. For this purpose, we need to check all the conditions of the theorem. First, we recall:

$$\tilde{X} := X \oplus X_h \text{ and } Y = \mathbf{H}^{-1}(\Omega), \tag{4.21}$$

and define a linear operator $T \in \mathcal{L}(Y; \tilde{X})$ as follows:
for given $\mathbf{f} \in Y$, $T\mathbf{f} = (\mathbf{u}_s, p_s) \in \tilde{X}$ is the solution of the Stokes problem:

$$\begin{aligned} -\nu\Delta\,\mathbf{u}_s + \mathbf{grad}\,p_s &= \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div}\mathbf{u}_s &= 0 \quad \text{in } \Omega \\ \mathbf{u}_s &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{4.22}$$

Next define the $\mathcal{C}^2$ mapping $\tilde{G} : R_+ \times \tilde{X} \to Y$ by

$$\tilde{G}(\lambda, u) = \lambda(\sum_{i=1}^{2} u_i D_i \mathbf{u} - \mathbf{f}), \quad u = (\mathbf{u}, p) \in \tilde{X}, \tag{4.23}$$

and we see that

$$D_u\tilde{G}(\lambda, u) \cdot w = \lambda \sum_{i=1}^{2}(u_i D_i \mathbf{w} + w_i D_i \mathbf{u}), \quad w = (\mathbf{w}, r) \in \tilde{X}. \tag{4.24}$$

By the fundamental Sobolev Imbedding Theorem, the imbedding of H_0^1 into $L^p(\Omega)$ is compact for $p < 6$. Therefore for $\mathbf{u}$ and $\mathbf{w}$ in $\mathbf{H}_0^1$, we have

$$\sum_{i=1}^{2}(u_i D_i \mathbf{w} + w_i D_i \mathbf{u}) \in (L^{3/2}(\Omega))^2.$$

So, we can choose

$$Z = (L^{3/2}(\Omega))^2 \hookrightarrow Y$$

with a compact imbedding which satisfies (4.7).

Now, let $T_h \in \mathcal{L}(Y; X_h)$ be the approximate linear operator defined by: for given $\mathbf{f} \in Y$, $(\mathbf{u}_{s,h}, p_{s,h}) = T_h\mathbf{f} \in X_h$ is the solution of

$$\begin{aligned} \nu(\nabla\mathbf{u}_{s,h}, \nabla\mathbf{v}_h) - (p_{s,h}, \operatorname{div}\,\mathbf{v}_h) = (\mathbf{f}, \gamma v_h), \quad \forall \mathbf{v}_h \in \mathbf{H}_0^h, \\ (r_h, \operatorname{div}\,\mathbf{u}_h) = 0, \quad \forall r_h \in L_0^h. \end{aligned} \tag{4.25}$$

As it is proved for the Stokes problem in [11], under the hypotheses $\mathbf{H}1, H2$ and $\mathbf{H}3$,

$$\lim_{h\to 0} \{|\mathbf{u}_{s,h} - \mathbf{u}_s|_1 + \|p_{s,h} - p_s\|_0\} = 0, \tag{4.26}$$

i.e.

$$\lim_{h\to 0} \|(T_h - T)\mathbf{f}\|_{\tilde{X}} = 0, \quad \forall\, \mathbf{f} \in Y.$$

Moreover, when $(\mathbf{u}_s, p_s)$ belongs to $\mathbf{H}^2(\Omega) \times H^1(\Omega)$ we have the error bound by the covolume analysis for the Stokes case [6]:

$$|\mathbf{u}_{s,h} - \mathbf{u}_s|_1 + \|p_{s,h} - p_s\|_0 \le Ch(\|\mathbf{u}_s\|_2 + \|p_s\|_1), \tag{4.27}$$

i.e

$$\|(T_h - T)\mathbf{f}\|_{\tilde{X}} \le Ch\|T\mathbf{f}\|_{\mathbf{H}^2(\Omega)\times H^1(\Omega)}.$$

Therefore the compactness of the imbedding of Z into Y together with (4.26) imply that

$$\lim_{h\to 0} \|(T_h - T)\|_{\mathcal{L}(Z;\tilde{X})} = 0.$$

Thus (4.8) and (4.9) hold.
Since

$$A(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v}_h) = \nu(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) + \big(\sum_{i=1}^{2} u_{hi}D_i\mathbf{u}_h, \gamma\mathbf{v}_h\big)$$

for the Navier-Stokes equations,(4.15) can be expressed as follows:

$$(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h) - (1/\nu)(p_h, \text{div } \mathbf{v}_h) = (1/\nu)\bigg(\mathbf{f} - \sum_{i=1}^{2} u_{hi}D_i\mathbf{u}_h, \gamma\mathbf{v}_h\bigg) \quad \forall\mathbf{v}_h \in \mathbf{H}_0^h,$$

$$(q_h, \text{div } \mathbf{u}_h) = 0 \quad \forall q_h \in L_0^h.$$

By (4.25), $u_h := (\mathbf{u}_h, (1/\nu)p_h)$ satisfies:

$$u_h = T_h\bigg[(1/\nu)\big(\mathbf{f} - \sum_{i=1}^{2} u_{hi}D_i\mathbf{u}_h\big)\bigg] = -T_h\tilde{G}(1/\nu, u_h).$$

Thus, an equivalent form of problem (4.15) is: find $u_h \in X_h$ solution of

$$F_h(\lambda, u_h) = u_h + T_h\tilde{G}(\lambda, u_h) = 0 \quad \text{with } \lambda = 1/\nu.$$

Thus, we can apply the conclusion of Theorem 4.1:
for $h \le h_0$ sufficiently small there exists a unique branch $\{(\lambda, u_h(\lambda) = (\mathbf{u}_h(\lambda), \lambda p_h(\lambda))); \lambda \in \Lambda\}$ of nonsingular solutions of (4.15) which is equivalent to the equation

$$u_h + T_h\tilde{G}(\lambda, u_h) = 0, \quad \forall\lambda \in \Lambda,$$

and a real number $a > 0$, independent of h, such that:

$$\|u_h(\lambda) - u(\lambda)\|_{\tilde{X}} \le a \quad \forall\lambda \in \Lambda.$$

Furthermore, (4.12) implies that

$$|u_h(\lambda) - u(\lambda)|_1 + |\lambda|\|p_h(\lambda) - p(\lambda)\|_0 \le K\|(T_h - T)\tilde{G}(\lambda, u(\lambda))\|_{\tilde{X}}.$$

Hence (4.19) follows from (4.26). Since

$$u(\lambda) = (\mathbf{u}(\lambda), \lambda p(\lambda)) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$$

is the solution of the Stokes system:

$$u(\lambda) = -T\tilde{G}(\lambda, u(\lambda)),$$

the error estimate for covolume scheme with $\tilde{G}(\lambda, u_h(\lambda))$ as right hand side gives:

$$\|(T_h - T)\tilde{G}(\lambda, u(\lambda))\|_{\tilde{X}} \leq Ch\{\|\mathbf{u}(\lambda)\|_2 + \|p(\lambda)\|_1\}.$$

Thus (4.20) is satisfied by the continuity of the mapping $\lambda \to u(\lambda)$ from Λ into $\mathbf{H}^2(\Omega) \times H^(\Omega)$. □

5 Numerical examples

Since the discretization of the Navier-Stokes equations (3.13) gives the following nonlinear system of equations, we need to use iterative methods to solve it.

$$\begin{aligned} \mathbf{Au} + \mathbf{C}(\mathbf{u}) + \mathbf{Bp} &= \mathbf{F} \\ \mathbf{B}^T\mathbf{u} &= 0 \end{aligned} \tag{5.1}$$

where

$$(\mathbf{u})^T = [\mathbf{u}_1^T \;\; \mathbf{u}_2^T], \quad \mathbf{u}_i^T = [u_i^1 \;\cdots\; u_i^N], \quad i = 1, 2,$$

for N nodal velocity and

$$\mathbf{p}^T = [p_1 \;\cdots\; p_L],$$

where L is the number of elements in discretization.
Matrices $\mathbf{A}$ and $\mathbf{B}$ are $2N \times 2N$ and $2N \times L$, respectively. The force vector $\mathbf{F}$ is $2N \times 1$ and $\mathbf{C}(u)$ is nonlinear term. Just for simplicity, we assume $\nu = 1$.

In our paper, we use the Picard iteration method. The method for the discrete equations (5.1) is described as follows. Suppose we are given $(\mathbf{u}_0, \mathbf{p}_0)$. First, we linearize the nonlinear equations based on the previous iteration solutions. Next, solve the resulting system of linear equations for $\mathbf{u}_k, \mathbf{p}_k$ at iterate k:

$$\begin{aligned} \mathbf{Au}_k + \mathbf{Bp}_k &= \mathbf{F}_k - \mathbf{C}(\mathbf{u}_{k-1}) \\ \mathbf{B}^T\mathbf{u}_k &= 0. \end{aligned} \tag{5.2}$$

where $k = 1, 2, 3, ...$
In each iteration, we use the Uzawa algorithm with Conjugate direction to solve the system (5.2).

Examples. For numerical computations, we have chosen the following test problems on the unit rectangular domain $\bar{\Omega} = [0, 1] \times [0, 1]$ with exact solution

$$\begin{aligned} u_1(x, y) &= Cx^2(x-1)^2 y(y-1)(2y-1) \\ u_2(x, y) &= -Cx(x-1)(2x-1)y^2(y-1)^2 \\ p(x, y) &= 2(x - \frac{1}{2})(y - \frac{1}{2}) \end{aligned}$$

where C is a constant.
We compared the results of the nonconforming covolume method with those of the nonconforming finite element method which are shown in Table.1 and Table.2.

Table.1 Errors and orders of convergence for the triangular meshes when $C = 1$.

	Nonconforming covolume				Nonconforming FEM			
	$\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}$		$\|p-p_h\|_{L_2(\Omega)}$		$\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}$		$\|p-p_h\|_{L_2(\Omega)}$	
n	error	order	error	order	error	order	error	order
8	1.428e-03		3.261e-02		1.429e-03		3.257e-02	
16	4.416e-04	1.69	1.441e-02	1.18	4.423e-04	1.69	1.441e-02	1.18
32	1.190e-04	1.89	6.536e-03	1.14	1.192e-04	1.89	6.532e-03	1.14
64	3.052e-05	1.96	3.109e-03	1.07	3.057e-05	1.96	3.109e-03	1.07
128	7.69e-06	1.99	1.524e-03	1.03	7.70e-06	1.99	1.524e-03	1.03

Table.2 Errors and orders of convergence for the triangular meshes when $C = 4$.

	Nonconforming covolume				Nonconforming FEM			
	$\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}$		$\|p-p_h\|_{L_2(\Omega)}$		$\|\mathbf{u}-\mathbf{u}_h\|_{L_2(\Omega)}$		$\|p-p_h\|_{L_2(\Omega)}$	
n	error	order	error	order	error	order	error	order
8	1.539e-03		3.388e-02		1.596e-03		3.364e-02	
16	4.681e-04	1.72	1.514e-02	1.16	4.822e-04	1.73	1.498e-02	1.17
32	1.255e-04	1.90	7.085e-03	1.10	1.290e-04	1.90	6.846e-03	1.13
64	3.240e-05	1.95	3.436e-03	1.04	3.297e-05	1.97	3.285e-03	1.06
128	8.23e-06	1.98	1.685e-03	1.03	8.34e-06	1.98	1.640e-03	1.00

6 Conclusion

- We introduced a new covolume method for the stationary Navier-Stokes equations and proved the convergence of the covolume approximation using an abstract theory for the nonlinear problem.
- Efficiency of the proposed method has been tested on a number of test problems. Numerical experiments show that the velocity errors with the covolume method are marginally better than those of the nonconforming finite element method.

- Although we have used the original form of the Navier-Stokes equations for the discretization method, the divergence form of the Navier Stokes equations can also be used to derive another scheme and all the results obtained here hold similarly.

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