On The Fine Spectrum of Fibonacci-type Operators on Banach Space ℓ_1

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Abstract

In this paper, we examine the fine spectrum of Lucas and Lucas like operators in Banach space ℓ_1.

Keywords: Banach space, linear operator, dual space, bounded operator, spectrum.

1 Preliminaries and notations.

By ω, we shall denote the space of all real or complex valued sequences. Any vector subspace of ω is called a sequence space. Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk}x_k \tag{1}$$

Let X be a Banach space and $T : X \to X$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in X : y = Tx, x \in X\}. \tag{2}$$

By $B(X)$, we denote the set of all bounded linear operators on X into itself. If $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual space X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.
Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T we associate the operator

$$T_\lambda = T - \lambda I, \tag{3}$$

where λ is a complex number and I is the identity operator on $D(T)$. If T_λ has an inverse which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1}, \tag{4}$$

and call it the *resolvent* operator of T. Many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. The boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ's the domain of T_λ^{-1} is dense in X, to name just a few aspects.

Definition 1.1. Let $X \neq \{\theta\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. A *regular value* λ of T is a complex number such that :
(R1) T_λ^{-1} exists,
(R2) T_λ^{-1} is bounded,
(R3) T_λ^{-1} is defined on a set which is dense in X.

The *resolvent set* of T, denoted by $\rho(T, X)$, is the set of all regular values λ of T. Its complement $\sigma(T, X) = \mathbb{C} \setminus \rho(T, X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:
The *point(discrete) spectrum* $\sigma_p(T, X)$ is the set such that T_λ^{-1} does not exist. Any such $\lambda \in \sigma_p(T, X)$ is called an *eigenvalue* of T.
The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_λ^{-1} exists and satisfies (R3) but not (R2), that is, T_λ^{-1} is unbounded.
The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_λ^{-1} exists (and may be bounded or not) but does not satisfy (R3), that is, the domain of T_λ^{-1} is not dense in X.
Hence if $(T - \lambda I)x = \theta$ for some $x \neq \theta$, then $\lambda \in \sigma_p(T, X)$, by definition, that is, λ is an eigenvalue of T. The vector x is then called an *eigenvector* of T corresponding to the eigenvalue λ.

Lemma 1.2. [1] *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Lemma 1.3. [1] The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_∞ norms of the columns of A is bounded.

2 The Spectrum of the Lucas and Lucas-like Operators on the Sequence Space ℓ_1

Let the linear operator $L_n : \ell_1 \to \ell_1$ be defined by

$$(x_1, x_2, x_3, \dots) \to \left(x_n + x_{n+1} + 2 \sum_{k=n+2}^{\infty} x_k, 2x_1, 2x_2, \dots \right) \tag{5}$$

Each L_n is bounded and its matrix representation in the standard basis is

$$\mathbf{L}_n = \begin{pmatrix} \overbrace{0 \quad \dots \quad 0}^{n-1} & 1 & 1 & 2 & 2 & \dots & & & \\ 2 \quad 0 \quad 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \\ 0 \quad 2 \quad 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \\ 0 \quad 0 \quad 2 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & \\ 0 \quad 0 \quad 0 & 2 & 0 & 0 & 0 & 0 & \dots & & \\ \vdots \quad \vdots \quad \vdots & \vdots & \ddots & & & & & & \end{pmatrix} \tag{6}$$

In this paper, our purpose is to investigate the fine spectrum of Lucas operator and Lucas-like operator in the space ℓ_1

Theorem 2.1.

$$\begin{aligned}
\sigma_p(L_n,\ell_1) &= \{\lambda\in\mathbb{C}:\lambda^{n+2}-2\lambda^{n+1}-2^{n-1}\lambda^2-2^{n+1}=0, |\lambda|>2\} &(7)\\
\rho(L_n,\ell_1) &= \{\lambda\in\mathbb{C}:\lambda^{n+2}-2\lambda^{n+1}-2^{n-1}\lambda^2-2^{n+1}\neq 0, |\lambda|>2\} &(8)\\
\sigma_r(L_n,\ell_1) &= \{\lambda\in\mathbb{C}:|\lambda|\leq 2,\lambda\neq 2\} &(9)\\
\sigma_c(L_n,\ell_1) &= \{2\}. &(10)
\end{aligned}$$

Proof. Our proof is divided into following four steps
Step 1. Now we have to solve following equation

$$(\lambda I-L_n)x=0, x\neq 0 \tag{11}$$

The equation (11) can be written as

$$\begin{aligned}
\lambda x_1 &= x_n+x_{n+1}+2\sum_{k=n+2}^{\infty}x_k\\
\lambda x_2 &= 2x_1\\
\lambda x_3 &= 2x_2\\
&\vdots\\
\lambda x_{k+1} &= 2x_k\\
&\vdots
\end{aligned} \tag{12}$$

Since $\lambda=0$ implies $x=0$, zero is not an element $\sigma_p(L_n,l_1)$. Then we have $\lambda\neq 0$, by applying (12) recursively, we have

$$x_{k+1}=\frac{2}{\lambda}x_k=\left(\frac{2}{\lambda}\right)^2x_{k-1}=\cdots=\left(\frac{2}{\lambda}\right)^kx_1, \qquad k\geq 1. \tag{13}$$

Hence

$$x=x_1\begin{pmatrix}1 & \frac{2}{\lambda} & \left(\frac{2}{\lambda}\right)^2 & \ldots\left(\frac{2}{\lambda}\right)^k\ldots\end{pmatrix}^t \tag{14}$$

and

$$\|x\|_1=|x_1|\sum_{k=0}^{\infty}\frac{2^k}{|\lambda|^k}. \tag{15}$$

If $|\lambda|\leq 2$, then $\|x\|_1=\infty$, so $x\notin\ell_1$. If $|\lambda|>2$, then $\|x\|_1=|x_1|\frac{|\lambda|}{|\lambda|-2}$. Inserting (13) in to the first equality of (12) gives

$$\begin{aligned}
0 &= \lambda x_1 - \left(\frac{2}{\lambda}\right)^{n-1} x_1 - \left(\frac{2}{\lambda}\right)^{n} x_1 - 2\sum_{k=n+2}^{\infty}\left(\frac{2}{\lambda}\right)^{k-1} x_1 \\
&= x_1\left(\lambda - \frac{2^{n-1}}{\lambda^{n-1}} - \frac{2^n}{\lambda^n} - 2\left(\frac{2}{\lambda}\right)^{n+1}\left[1+\left(\frac{2}{\lambda}\right)+\left(\frac{2}{\lambda}\right)^2+\ldots\right]\right) \\
&= x_1\left(\frac{\lambda^{n+1}-2^{n-1}\lambda-2^n}{\lambda^n} - 2\left(\frac{2}{\lambda}\right)^{n+1}\cdot\frac{1}{1-\frac{2}{\lambda}}\right) \\
&= x_1\frac{\lambda^{n+2}-2\lambda^{n+1}-2^{n-1}\lambda^2-2^{n+1}}{\lambda^n(\lambda-2)}.
\end{aligned}$$

Since $x_1 \neq 0$, we conclude that $\sigma_p(L_n, \ell_1)$ consists of those roots of the polynomial

$$P_{n+2}(\lambda) = \lambda^{n+2} - 2\lambda^{n+1} - 2^{n-1}\lambda^2 - 2^{n+1} \tag{16}$$

for which $|\lambda| > 2$, as stated in (12). Since $P_{n+2}(2) = -2^{n+2} < 0$, for all $n \in \mathbb{N}$, $P_{n+2}(\frac{5}{2}) = 5^{n+1} - 9\cdot 4^n > 0$ for all positive integers such that $n \geq 3$ and $P'_{n+2}(\lambda) > 0$ for $\lambda \in \mathbb{R}, \lambda \geq 2$, that is, P_{n+2} is strictly increasing for $\lambda > 2$, from here we have the polynomial P_{n+2} has exactly one real root larger than 2, we conclude that L_n has exactly one real eigenvalue larger than two. Let us denote this eigenvalue by $\lambda_{\max}(L_n)$. By Ostrovsky's theorem (V.V.Prasolov [3, Theorem 1.1.4,p.3]),$\lambda_{\max}(L_n)$ is the unique positive root of $P_{n+2}(\lambda)$ and the absolute values of all other roots are strictly smaller. Consequently, all other eigenvalues of L_n are in absolute value strictly smaller than $\lambda_{\max}(L_n)$ which, in turn, implies

$$r_\sigma(L_n) = \lambda_{\max}(L_n) \tag{17}$$

Step 2. Now we have to solve following equation,

$$(\lambda I - L_n)x = y, x \neq 0. \tag{18}$$

The equation (18) can be written as

$$\begin{aligned}
\lambda x_1 - (x_n + x_{n+1}) - 2\sum_{k=n+2}^{\infty} x_k &= y_1 \\
\lambda x_2 - 2x_1 &= y_2 \\
\lambda x_3 - 2x_2 &= y_3 \\
&\vdots \\
\lambda x_{k+1} - 2x_k &= y_{k+1} \\
&\vdots
\end{aligned} \tag{19}$$

Using (19), we have

$$a = \frac{2x_n}{\lambda} + \frac{2a}{\lambda} + \frac{b}{\lambda}$$

and

$$y_1 = \lambda x_1 - x_n + x_{n+1} - 2a,$$

where

$$a = \sum_{k=n+1}^{\infty} x_k, \qquad b = \sum_{k=n+1}^{\infty} y_k$$

After rearranging, we have

$$a = \frac{1}{\lambda - 2}(2x_n + b).$$

Thus,

$$y_1 = \lambda x_1 + x_{n+1} - \frac{(\lambda+2)x_n + 2b}{\lambda - 2} \tag{20}$$

By recursively applying (19), we have

$$\begin{aligned}
x_2 &= \frac{2}{\lambda}x_1 + \frac{1}{\lambda}y_2 \\
x_3 &= \left(\frac{2}{\lambda}\right)^2 x_1 + \frac{1}{2}\left(\frac{2}{\lambda}\right)^2 y_2 + \frac{1}{2}\left(\frac{2}{\lambda}\right) y_3 \\
&\vdots \\
x_{k+1} &= \left(\frac{2}{\lambda}\right)^k x_1 + \frac{1}{2}\left[\left(\frac{2}{\lambda}\right)^k y_2 + \left(\frac{2}{\lambda}\right)^{k-1} y_3 + \cdots + \left(\frac{2}{\lambda}\right) y_{k+1}\right] \qquad (21) \\
&\vdots
\end{aligned}$$

Inserting x_n and x_{n+1} into (20) gives

$$\begin{aligned}
y_1 =& \lambda x_1 + \left(\frac{2}{\lambda}\right)^n x_1 + \frac{1}{2}\left[\left(\frac{2}{\lambda}\right)^n y_2 + \left(\frac{2}{\lambda}\right)^{n-1} y_3 + \cdots + \left(\frac{2}{\lambda}\right) y_{n+1}\right] \\
&- \left(\frac{\lambda+2}{\lambda-2}\right)\left(\left(\frac{2}{\lambda}\right)^{n-1} x_1 + \frac{1}{2}\left[\left(\frac{2}{\lambda}\right)^{n-1} y_2 + \left(\frac{2}{\lambda}\right)^{n-2} y_3 + \cdots + \left(\frac{2}{\lambda}\right) y_n\right]\right) \\
&- \frac{2b}{\lambda-2}
\end{aligned}$$

and solving for x_1 gives

$$\begin{aligned}
x_1 =& \frac{1}{P_{n+2}(\lambda)}\big(\lambda^n(\lambda-2)y_1 + (\lambda^2+4)\left[2^{n-2}\lambda^0 y_2 + 2^{n-3}\lambda^1 y_3 + \cdots + 2^0\lambda^{n-2}y_n\right] \\
&+ (2\lambda^{n-1} - \lambda^n)y_{n+1} + 2\lambda^n b\big)
\end{aligned}$$

By inserting this into (21) we have

$$x = (\lambda I - L_n)^{-1}y = \frac{1}{P_{n+2}(\lambda)}\,(A+B)\,y,$$

where the matrix representations of A and B are given by

$$\mathbf{A}=\begin{pmatrix} \lambda^n(\lambda-2) & (\lambda^2+4)2^{n-2} & \dots & (\lambda^2+4)\lambda^{n-2} & \lambda^n+2\lambda^{n-1} & 2\lambda^n & \dots \\ 2\lambda^{n-1}(\lambda-2) & \frac{(\lambda^2+4)2^{n-1}}{\lambda} & \dots & (\lambda^2+4)\lambda^{n-3} & \frac{2(\lambda^n+2\lambda^{n-1})}{\lambda} & 2^2\lambda^{n-1} & \dots \\ 2^2\lambda^{n-2}(\lambda-2) & \frac{(\lambda^2+4)2^n}{\lambda^2} & \dots & (\lambda^2+4)\lambda^{n-4} & \frac{2^2(\lambda^n+2\lambda^{n-1})}{\lambda^2} & 2^3\lambda^{n-2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 2^{n-1}\lambda(\lambda-2) & \frac{(\lambda^2+4)2^{2n-3}}{\lambda^{n-2}} & \dots & (\lambda^2+4)\lambda^{-1} & \frac{2^n(\lambda^n+2\lambda^{n-1})}{\lambda^n} & 2^n\lambda & \dots \\ 2^n(\lambda-2) & \frac{(\lambda^2+4)2^{2n-2}}{\lambda^{n-1}} & \dots & (\lambda^2+4)\lambda^{-2} & \frac{2^{n+1}(\lambda^n+2\lambda^{n-1})}{\lambda^{n+1}} & 2^{n+1} & \dots \\ \frac{2^{n+1}(\lambda-2)}{\lambda} & \frac{(\lambda^2+4)2^{2n-1}}{\lambda^n} & \dots & (\lambda^2+4)\lambda^{-3} & \frac{2^{n+2}(\lambda^n+2\lambda^{n-1})}{\lambda^{n+2}} & \frac{2^{n+2}}{\lambda} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\mathbf{B}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\lambda} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2}\left(\frac{2}{\lambda}\right)^2 & \frac{1}{\lambda} & 0 & 0 & \dots \\ 0 & \frac{1}{2}\left(\frac{2}{\lambda}\right)^3 & \frac{1}{2}\left(\frac{2}{\lambda}\right)^2 & \frac{1}{\lambda} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

Obviously, for $|\lambda| > 2$ by the lemma1.2 we have $\|A\|_1 < \infty$ and $\|B\|_1 < \infty$. Thus for $|\lambda| > 2$ and λ not being the root of $\lambda^{n+2} - 2\lambda^{n+1} - 2^{n-1}\lambda^2 - 2^{n+1}$, the operator $\lambda I - L_n$ has a bounded inverse, so the resolvent set of L_n is given by (8).

It should be noted that the adjoint operator L_n^* of L_n is an operator on the dual space of ℓ_1 which is isometrically isomorphic to Banach space ℓ_∞. Hence we have

$$\sigma_r(L_n, \ell_1) \subseteq \sigma_p(L_n^*, \ell_\infty) \tag{22}$$

Step 3. In this step we would prove that

$$\sigma_p(L_n^*, \ell_\infty) = \{\lambda \in \mathbb{C} : |\lambda| \leq 2, \lambda \neq 2\} \tag{23}$$

$$(\lambda I - L_n^*)x = 0, \quad x \neq 0, \quad \|x\|_\infty < \infty \tag{24}$$

This is equivalent to

$$
\begin{aligned}
x_2 &= \left(\frac{\lambda}{2}\right) x_1 \\
x_3 &= \left(\frac{\lambda}{2}\right) x_2 = \left(\frac{\lambda}{2}\right)^2 x_1 \\
&\vdots \\
x_{n-1} &= \left(\frac{\lambda}{2}\right) x_{n-2} = \left(\frac{\lambda}{2}\right)^{n-2} x_1 \\
x_n &= \left[\left(\frac{\lambda}{2}\right)^{n-1} - \frac{1}{2}\right] x_1 \\
x_{n+1} &= \left[\left(\frac{\lambda}{2}\right)^{n} - \frac{\lambda}{2^2} - 1\right] x_1 \\
x_{n+2} &= \left[\left(\frac{\lambda}{2}\right)^{n+1} - \frac{1}{2}\left(\frac{\lambda}{2}\right)^2 - \left(\frac{\lambda}{2}\right) - 1\right] x_1 \\
&\vdots \\
x_{n+k} &= \left[\left(\frac{\lambda}{2}\right)^{n+k-1} - \frac{1}{2}\left(\frac{\lambda}{2}\right)^k - \left(\frac{\lambda}{2}\right)^{k-1} - \left(\frac{\lambda}{2}\right)^{k-2} - \cdots - \left(\frac{\lambda}{2}\right)^2 - \left(\frac{\lambda}{2}\right) - 1\right] x_1 \\
&\vdots
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
x_{n+k} &= \left[\left(\frac{\lambda}{2}\right)^{n+k-1} + \left(\frac{\lambda}{2}\right)^k - \frac{\left(\frac{\lambda}{2}\right)^{k+1} - 1}{\left(\frac{\lambda}{2}\right) - 1}\right] x_1 \\
&= \left[\mu^{n+k-1} + \mu^k - \frac{\mu^{k+1} - 1}{\mu - 1}\right] x_1
\end{aligned}
$$

where $\frac{\lambda}{2} = \mu$, so if $|\lambda \leq 2|$ then $|\mu| \leq 1$. For $|\mu| \leq 1, \mu \neq 1$ we have

$$|x_k| < \left(2 + \frac{2}{|\mu - 1|}\right) |x_1|.$$

Thus by lemma1.3 $\|x\|_\infty < \infty$. For $\lambda = 2$ we have

$$\begin{aligned}
x_2 &= x_1, \\
x_3 &= x_1, \\
&\vdots \\
x_{n-1} &= x_1, \\
x_n &= \frac{1}{2} x_1, \\
x_{n+1} &= -\frac{1}{2} x_1, \\
x_{n+2} &= -\frac{3}{2} x_1, \\
&\vdots \\
x_{n+k} &= -\frac{2k-1}{2} x_1, \\
&\vdots
\end{aligned}$$

so $\|x\|_\infty = \infty$. We conclude that the point spectrum of L_n^* is given by (23). This, in turn with (22) , implies (9) as described before. *Step 4.*

$$\sigma_c(L_n, \ell_1) = \mathbb{C} \setminus (\rho(L_n, \ell_1) \cup \sigma_p(L_n, \ell_1) \cup \sigma_r(L_n, \ell_1)) = \{2\}$$

□

Let the linear operator $T_n : \ell_1 \to \ell_1$ be defined by

$$(x_1, x_2, x_3, \dots) \to \left(x_n + 2 \sum_{k=n+1}^{\infty} x_k, 2x_1, 2x_2, \dots \right) \tag{25}$$

Each T_n is bounded and its matrix representation in the standard basis is

$$\mathbf{T}_n = \begin{pmatrix}
\overbrace{0 \quad \dots \quad 0}^{n-1} & 1 & 2 & 2 & 2 & \dots & & \\
2 \quad 0 \quad\;\; 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\
0 \quad 2 \quad\;\; 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\
0 \quad 0 \quad\;\; 2 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\
0 \quad 0 \quad\;\; 0 & 2 & 0 & 0 & 0 & 0 & \dots & \\
\vdots \quad \vdots \quad\;\; \vdots & \vdots & \ddots & & & & &
\end{pmatrix} \tag{26}$$

By the similarly method above Theorem2.1 we can prove following theorem:

Theorem 2.2.

$$\begin{aligned}
\sigma_p(T_n, \ell_1) &= \{\lambda \in \mathbb{C} : \lambda^{n+1} - 2\lambda^n - 2^{n-1}\lambda - 2^n = 0, |\lambda| > 2\} && (27) \\
\rho(T_n, \ell_1) &= \{\lambda \in \mathbb{C} : \lambda^{n+1} - 2\lambda^n - 2^{n-1}\lambda - 2^n \neq 0, |\lambda| > 2\} && (28) \\
\sigma_r(T_n, \ell_1) &= \{\lambda \in \mathbb{C} : |\lambda| \leq 2, \lambda \neq 2\} && (29) \\
\sigma_c(T_n, \ell_1) &= \{2\}. && (30)
\end{aligned}$$

Let the linear operator $Q_n : \ell_1 \to \ell_1$ be defined by

$$(x_1, x_2, x_3, \dots) \to \left(\sum_{k=1}^{\infty} x_k, nx_1, nx_2, \dots \right) \tag{31}$$

Each Q_n is bounded and its matrix representation in the standard basis is

$$\mathbf{Q}_n = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ n & 0 & 0 & 0 & \dots \\ 0 & n & 0 & 0 & \dots \\ 0 & 0 & n & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{32}$$

Theorem 2.3.

$$\sigma_p(Q_n, \ell_1) = \{\lambda = n+1\} \tag{33}$$
$$\rho(Q_n, \ell_1) = \{\lambda \in \mathbb{C} : \lambda^2 - (n+1)\lambda \neq 0, |\lambda| > n\} \tag{34}$$
$$\sigma_r(Q_n, \ell_1) = \{\lambda \in \mathbb{C} : |\lambda| \leq n, \lambda \neq n\} \tag{35}$$
$$\sigma_c(Q_n, \ell_1) = \{n\}. \tag{36}$$

Proof. Our proof is divided into following four steps
Step 1. Now we have to solve following equation

$$(\lambda I - Q_n)x = 0, x \neq 0 \tag{37}$$

The equation (37) can be written as

$$\begin{aligned} \lambda x_1 &= \sum_{k=1}^{\infty} x_k \\ \lambda x_2 &= nx_1 \\ \lambda x_3 &= nx_2 \\ &\vdots \\ \lambda x_{k+1} &= nx_k \\ &\vdots \end{aligned} \tag{38}$$

Since $\lambda = 0$ implies $x = 0$, zero is not an element $\sigma_p(Q_n, l_1)$. Then we have $\lambda \neq 0$, by applying (38) recursively, we have

$$x_{k+1} = \left(\frac{n}{\lambda}\right) x_k = \left(\frac{n}{\lambda}\right)^2 x_{k-1} = \dots = \left(\frac{n}{\lambda}\right)^k x_1, \qquad k \geq 1. \tag{39}$$

Hence

$$x = x_1 \begin{pmatrix} 1 & \frac{n}{\lambda} & \left(\frac{n}{\lambda}\right)^2 & \dots \left(\frac{n}{\lambda}\right)^k \dots \end{pmatrix}^t \tag{40}$$

and

$$\|x\|_1 = |x_1| \sum_{k=0}^{\infty} \frac{n^k}{|\lambda|^k}. \tag{41}$$

If $|\lambda| \leq n$, then $\|x\|_1 = \infty$, so $x \notin \ell_1$. If $|\lambda| > n$, then $\|x\|_1 = |x_1| \frac{|\lambda|}{|\lambda|-n}$. Inserting (39) in to the first equality of (38) gives

$$\begin{aligned}
0 &= \lambda x_1 - x_1 - \left(\frac{n}{\lambda}\right) x_1 - \left(\frac{n}{\lambda}\right)^2 x_1 - \left(\frac{n}{\lambda}\right)^3 x_1 - \dots \\
&= x_1 \left[\lambda - 1 - \frac{n}{\lambda}\left(1 + \frac{n}{\lambda} + \left(\frac{n}{\lambda}\right)^2 + \left(\frac{n}{\lambda}\right)^3 + \dots\right)\right] \\
&= x_1 \left[\lambda - 1 - \frac{n}{\lambda} \cdot \frac{1}{1 - \frac{n}{\lambda}}\right] \\
&= x_1 \left[\frac{\lambda^2 - (n+1)\lambda}{\lambda - n}\right].
\end{aligned}$$

Since $x_1 \neq 0$, we conclude that $\sigma_p(Q_n, \ell_1)$ consists of just one point $n+1$. *Step 2.* Now we have to solve following equation,

$$(\lambda I - Q_n)x = y, x \neq 0. \tag{42}$$

The equation can be written as

$$\begin{aligned}
\lambda x_1 - \sum_{k=1}^{\infty} x_k &= y_1 \\
\lambda x_2 - n x_1 &= y_2 \\
\lambda x_3 - n x_2 &= y_3 \\
&\vdots \\
\lambda x_{k+1} - n x_k &= y_{k+1} \\
&\vdots
\end{aligned} \tag{43}$$

Using , we have

$$a = \frac{n}{\lambda}(a + x_1) + \frac{1}{\lambda} \cdot b$$

and

$$y_1 = \lambda x_1 - x_1 - a$$

,where

$$a = \sum_{k=2}^{\infty} x_k, \qquad b = \sum_{k=2}^{\infty} y_k$$

After rearranging, we have

$$a = \frac{n x_1 + b}{\lambda - n}.$$

Thus,

$$y_1 = \frac{\lambda^2 - (n+1)\lambda}{\lambda - n} x_1 - \frac{b}{\lambda - n}. \tag{44}$$

Hence, we have

$$x_1 = \frac{1}{\lambda^2 - (n+1)\lambda} \left[(\lambda - n) y_1 + b\right] \tag{45}$$

By recursively applying (43) and denoting that $f(\lambda) = \lambda^2 - (n+1)\lambda$, we have

$$\begin{aligned}
x_2 &= \frac{1}{f(\lambda)} \left(\frac{n}{\lambda}(\lambda - n) y_1 + \frac{n}{\lambda} b\right) + \frac{1}{\lambda} y_2 \\
x_3 &= \frac{1}{f(\lambda)} \left(\left(\frac{n}{\lambda}\right)^2 (\lambda - n) y_1 + \left(\frac{n}{\lambda}\right)^2 b\right) + \frac{n}{\lambda^2} y_2 + \frac{1}{\lambda} y_3 \\
&\vdots \\
x_{k+1} &= \frac{1}{f(\lambda)} \left(\left(\frac{n}{\lambda}\right)^k (\lambda - n) y_1 + \left(\frac{n}{\lambda}\right)^k b\right) + \frac{1}{n} \left(\left(\frac{n}{\lambda}\right)^k y_2 + \cdots + \left(\frac{n}{\lambda}\right) y_{k+1}\right) \\
&\vdots
\end{aligned} \tag{46}$$

Using (46) we have

$$x = (\lambda I - Q_n)^{-1} y = \left(\frac{1}{f(\lambda)} A + B\right) y,$$

where the matrix representations of A and B are given by

$$\mathbf{A} = \begin{pmatrix}
(\lambda - n) & 1 & 1 & 1 & \cdots \\
\frac{n}{\lambda}(\lambda - n) & \frac{n}{\lambda} & \frac{n}{\lambda} & \frac{n}{\lambda} & \cdots \\
\left(\frac{n}{\lambda}\right)^2 (\lambda - n) & \left(\frac{n}{\lambda}\right)^2 & \left(\frac{n}{\lambda}\right)^2 & \left(\frac{n}{\lambda}\right)^2 & \cdots \\
\left(\frac{n}{\lambda}\right)^3 (\lambda - n) & \left(\frac{n}{\lambda}\right)^3 & \left(\frac{n}{\lambda}\right)^3 & \left(\frac{n}{\lambda}\right)^3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix} \tag{47}$$

and

$$\mathbf{B} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{n}\left(\frac{n}{\lambda}\right) & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{n}\left(\frac{n}{\lambda}\right)^2 & \frac{1}{n}\left(\frac{n}{\lambda}\right) & 0 & 0 & \cdots \\
0 & \frac{1}{n}\left(\frac{n}{\lambda}\right)^3 & \frac{1}{n}\left(\frac{n}{\lambda}\right)^2 & \frac{1}{n}\left(\frac{n}{\lambda}\right) & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots &
\end{pmatrix} \tag{48}$$

Obviously, for $|\lambda| > n$ by lemma1.2 we have $\|A\|_1 < \infty$ and $\|B\|_1 < \infty$. Thus for $|\lambda| > n$ and λ not being the root of $f(\lambda)$, the operator $\lambda I - Q_n$ has a bounded inverse, so the resolvent set of Q_n is given by.

It should be noted that the adjoint operator Q_n^* of Q_n is an operator on the dual space of ℓ_1 which is isometrically isomorphic to Banach space ℓ_∞. Hence we have

$$\sigma_r(Q_n, \ell_1) \subseteq \sigma_p(Q_n^*, \ell_\infty) \tag{49}$$

Step 3. In this step we would prove that

$$\sigma_p(Q_n^*, \ell_\infty) = \{\lambda \in \mathbb{C} : |\lambda| \le n, \lambda \ne n\} \tag{50}$$

$$(\lambda I - Q_n^*)x = 0, \quad x \ne 0, \quad \|x\|_\infty < \infty \tag{51}$$

This is equivalent to

$$\begin{aligned}
x_2 &= \left(\frac{\lambda-1}{n}\right)x_1 \\
x_3 &= \frac{\lambda x_2 - x_1}{n} = \left[\frac{\lambda(\lambda-1)}{n^2} - \frac{1}{n}\right]x_1 \\
&\vdots \\
x_{k+1} &= \left[\frac{1}{n}\left(\frac{\lambda}{n}\right)^{k-1}(\lambda-1) - \left(\frac{\lambda}{n}\right)^{k-2} - \frac{1}{\lambda}\left(\left(\frac{\lambda}{n}\right)^{k-2} + \cdots + \left(\frac{\lambda}{n}\right)\right)\right]x_1 \\
&\vdots
\end{aligned}$$

Therefore,

$$\begin{aligned}
x_k &= \left(\frac{\lambda}{n}\right)^{k-2}\left(\frac{\lambda}{n} - \frac{1}{n}\right)x_1 - \left(\frac{\lambda}{n}\right)^{k-3}x_1 - \frac{\lambda^{k-3} - n^{k-3}}{(\lambda-n)n^{k-3}}x_1 \\
&= \left[\mu^{k-1} - \frac{\mu^{k-2}}{n} - \mu^{k-3} - \frac{\mu^{k-3}-1}{(\mu-1)n^{k-3}}\right]x_1
\end{aligned}$$

where $\frac{\lambda}{n} = \mu$, so if $|\lambda| \le n$ then $|\mu| \le 1$. For $|\mu| \le 1, \mu \ne 1$ we have

$$\forall k \ge 3 : |x_k| < \left(2 + \frac{1}{n} + \frac{2}{|\mu - 1|}\right)|x_1|.$$

By lemma1.3 $\|x\|_\infty < \infty$. For $\lambda = n$ we have

$$\begin{aligned}
x_2 &= \left(1 - \frac{1}{n}\right)x_1, \\
x_3 &= \left(1 - \frac{2}{n}\right)x_1, \\
x_4 &= -\frac{2}{n}x_1 \\
x_5 &= -\frac{3}{n}x_1 \\
&\vdots \\
x_{k+1} &= -\frac{k-1}{n}x_1, \\
&\vdots
\end{aligned}$$

Hence, $\|x\|_\infty = \infty \Rightarrow x \notin \ell_\infty$. We conclude that the point spectrum of Q_n^* is given by (50) . This, in turn with (49) , implies (35). *Step 4.*

$$\sigma_c(Q_n, \ell_1) = \mathbb{C} \setminus (\rho(Q_n, \ell_1) \cup \sigma_p(Q_n, \ell_1) \cup \sigma_r(Q_n, \ell_1)) = \{n\}$$

□

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