Semidefinite concave programming

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Abstract

We introduce so-called semidefinite concave programming or equivalently semidefinite convex maximization problem. We derive new global optimality conditions by generalizing Strekalovsky's theorem [1]. Based on the global optimality conditions, we construct an algorithm which generates a sequence of local maximizers converging to the global solution. Subproblems of the proposed algorithm are semidefinite linear programming.

Keywords. Semidefinite linear programming, global optimality conditions, semidefinite concave programming, algorithm, approximation set.

1 Introduction

Semidefinite linear programming can be regarded as an extension of linear programming and solves the following problem

$$\min\langle C, X\rangle_F$$
$$\langle A_j, X\rangle \leqslant b_j, j = 1, 2, .., s, \tag{1}$$
$$X \geqslant 0,$$

Here $X \in \mathbb{R}^{n\times n}$ is a matrix of variables and $A_j \in \mathbb{R}^{n\times n}, j = 1, 2, ..., s$. $X \geqslant 0$ is notation for " X is positive semidefinite ". $\langle\cdot,\cdot\rangle_F$ denotes Frobenius norm and $\|X\|_F = \sqrt{\langle A, A\rangle_F}$.

Semidefinite programming finds many applications in engineering and optimization [5]. Most interior-point methods for linear programming have been generalized to semidefinite convex programming [3, 4, 5]. There are many works devoted to the semidefinite convex programming problem but less attention so for has been paid to semidefinite concave programming or equivalently semidefinite convex maximization problem. Studying such problems is important not only from view of optimization theory but also economic applications. For instant, semidefinite convex utility maximization problem can be formulated as follows [1].

$$\max \mathrm{U}(X),$$

subject to:

$$\langle P, X \rangle \leqslant C,$$

$$X \geqslant 0,$$

where $\mathrm{U} : \mathbb{R}^{n\times n} \to \mathbb{R}$ is the consumer's convex utility function, X is a basket of goods (x_{ij} is amount of $j-th$ brand of $i-th$ good), P is a price matrix (p_{ij} price per a unit of x_{ij} good), C-is the total budget.
Aim of this paper is to develop theory and algorithms for semidefinite concave programming. The paper is organized as follows. Section 2 is devoted to formulation of semidefinite concave programming and its global optimality conditions. In Section 3, we consider an approximation of the level set of the objective function and its properties.

2 Problem Formulation and Optimality Conditions

Let X be matrices in $\mathbb{R}^{n\times n}$, and define a scalar matrix function as

$$f : \mathbb{R}^{n\times n} \to \mathbb{R}.$$

Definition 2.1. *Let $f(X)$ be a differentiable function of the matrix X. Then*

$$f'(X) = \left(\frac{\partial f(X)}{\partial x_{ij}} \right)_{n\times n}.$$

Introduce the Frobenius scalar product as

$$\langle X, Y \rangle_F = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ij}, \forall X, Y \in \mathbb{R}^{n\times n}.$$

If $f(\cdot)$ is differentiable, then it can be checked that

$$f(X+H) - f(X) = \langle f'(X), H\rangle_F + o(\|H\|_F).$$

Definition 2.2. *A set* $\mathbb{D} \subset \mathbb{R}^{n\times n}$ *is convex if* $\alpha X + (1-\alpha)Y \in \mathbb{D}$ *for all* $X, Y \in \mathbb{D}$ *and* $\alpha \in [0,1]$.

Definition 2.3. *The function* $f : \mathbb{D} \to \mathbb{R}$ *is said to be convex on* $\mathbb{D}$ *if* $f(\alpha X + (1-\alpha)Y) \leqslant \alpha f(X) + (1-\alpha) f(Y)$ *for all* $X, Y \in \mathbb{D}$ *and* $\alpha \in [0,1]$.

The well known property of a convex function [2] can be easily generalized as follows:

Lemma 2.1. *The function* $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ *is convex if and only if*

$$f(X) - f(Y) \geqslant \langle f'(Y), X - Y\rangle_F$$

for all $X, Y \in \mathbb{R}^{n\times n}$.

Consider the problem of maximizing a differentiable convex matrix function subject to constraints:

$$\max f(X) \tag{2.1}$$

subject to:

$$\langle A_j, X\rangle_F \leqslant b_j, j = 1, 2, .., s, \tag{2.2}$$

$$X \geqslant 0, \tag{2.3}$$

where $A_j \in \mathbb{R}^{n\times n}, j = 1, 2, ..., s$ and $X \geqslant 0$ are positive semidefinite matrices, $b_j \in \mathbb{R}$.

We call problem (2.1)-(2.3) as the semidefinite convex maximization problem or equivalently, semidefinite concave programming.

Denote by $\mathbb{D}$ constraints of the problem:

$$\mathbb{D} = \{X \in \mathbb{R}^{n\times n} | \langle A_j, X\rangle_F \leqslant b_j, j = 1, 2, .., s; X \geqslant 0\}.$$

Then problem (2.1)-(2.3) reduces to

$$\max_{X \in \mathbb{D}} f(X). \tag{2.4}$$

It can checked that the set $\mathbb{D}$ is convex. Problem (2.4) is nonconvex and belongs to a class of global optimization problems in Banach space.

Global Optimality Conditions

Introduce the level set $\mathrm{E}_{f(Z)}(f)$ of the function $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ at a point $Z \in \mathbb{R}^{n\times n}$:

$$\mathrm{E}_{f(Z)}(f) = \{Y \in \mathbb{R}^{n\times n} | f(Y) = f(Z)\}.$$

The global optimality condition for problem (2.4) can be formulated in the following assertion.

Theorem 2.1. *If $Z \in \mathbb{D}$ is a global solution to problem (2.4) then*

$$\langle f'(Y), X - Y\rangle_F \leqslant 0 \tag{2.5}$$

hold for all $Y \in \mathrm{E}_{f(Z)}(f)$ and $X \in \mathbb{D}$. If in addition,there exists a $\theta \subset \mathbb{R}^{n\times n}$ such that

$$f(Z) > f(\theta) > -\infty, \tag{2.6}$$

then condition (2.5) is sufficient for $Z \in \mathbb{D}$ to be a global solution to problem (2.4).

Proof. Necessity. Assume that Z is a solution to problem (2.1)-(2.3) and $Y \in \mathrm{E}_{f(Z)}(f)$ and $X \in \mathbb{D}$. Then by lemma 2.1, we have

$$0 \geqslant f(X) - f(Z) = f(X) - f(Y) \geqslant \langle f'(Y), X - Y\rangle_F$$

for all $Y \in \mathrm{E}_{f(Z)}(f)$ and $X \in \mathbb{D}$.

Sufficiency. Suppose, on the contrary, that Z is not a solution to problem (2.4),i.e, there exists an $U \in \mathbb{D}$ such that $f(U) > f(Z)$. Introduce the set $\mathrm{L}_{f(Z)}(f)$ as :

$$\mathrm{L}_{f(Z)}(f) = \{X \in \mathbb{R}^{n\times n} | f(X) \leqslant f(Z)\}.$$

Clearly, $\mathrm{L}_{f(Z)}(f) \neq \emptyset$. Moreover, the set $\mathrm{L}_{f(Z)}(f)$ is convex and closed. Let Y be the projection of U on $\mathrm{L}_{f(Z)}(f)$ such that

$$\|Y - U\|_F = \min_{X \in \mathrm{L}_{f(Z)}(f)} \|X - U\|_F.$$

Obviously,

$$\|Y - U\|_F > 0 \tag{2.7}$$

holds since $U \notin \mathrm{L}_{f(Z)}(f)$. The point Y can be considered as a solution of the convex minimization problem:

$$\min_{X \in \mathrm{L}_{f(Z)}(f)} \{g(X) = \frac{1}{2}\|X - U\|_F^2\}. \tag{2.8}$$

Applying the lagrange method to problem (2.8), we obtain the following optimality conditions at the point Y:

$$\begin{cases} \lambda_0 \geqslant 0, \quad \lambda \geqslant 0, \quad \lambda_0 + \lambda > 0, \\ \lambda_0 g'(Y) + \lambda f'(Y) = 0, \\ \lambda(f(Y) - f(Z)) = 0 \end{cases} \tag{2.9}$$

or equivalently,

$$\begin{cases} \lambda_0 \geqslant 0, \quad \lambda \geqslant 0, \quad \lambda_0 + \lambda > 0, \\ \lambda_0 (Y - U) + \lambda f'(Y) = 0, \\ \lambda(f(Y) - f(Z)) = 0. \end{cases} \tag{2.10}$$

If $\lambda = 0$, then (2.10) implies that $\lambda > 0, f(Y) = f(Z)$, and $f'(Y) = 0$. Since $f(\cdot)$ is convex, then Y is the global minimizer to problem $\min_{X \in \mathbb{R}^{n \times n}} f(X))$ which contradicts condition (2.6). If $\lambda = 0$, then we have $\lambda_0 > 0$, and $g'(Y) = Y - U = 0$ which also contradicts (2.7). So, without loss of generality, we can set $\lambda_0 = 1$ and $\lambda > 0$ in (2.10). Hence, we have

$$Y - U + \lambda f'(Y) = 0, \quad \lambda > 0.$$

From this, we can conclude that

$$\lambda f'(Y) = U - Y,$$

and

$$\lambda\langle f'(Y), U - Y\rangle_F = \|U - Y\|_F^2 > 0$$

which contradicts (2.5). Last contradiction implies that the assumption that Z is not a global solution to problem (2.4) must be false. This completes the proof.

Remark 2.1. *If we set $Y = Z$ in (2.5), we obtain the the well known local optimality condition [2]:*

$$\langle f'(Z), X - Z\rangle_F \leqslant 0, \quad \forall X \in \mathbb{D}.$$

Remark 2.2. *For a fixed $Y \in \mathrm{E}_{f(Z)}(f)$, checking condition (2.5) reduces to*

$$\max_{x\in\mathbb{D}}\langle f'(Y), X\rangle_F \leqslant \langle f'(Y), Y\rangle$$

or equivalently to semidefinite linear programming:

$$\max\langle f'(Y), X\rangle_F$$

subject to:

$$\langle A_j, X\rangle \leqslant b_j, j = 1, 2, ..., s,$$

$$X \geqslant 0.$$

Remark 2.3. *In order to conclude that a point $\widetilde{Z} \in \mathbb{D}$ is not a global solution to problem (2.4), we need to find a pair $(U, \widetilde{Y})$ such that*

$$\langle f'(\widetilde{Y}), U - \widetilde{Y}\rangle_F > 0, \widetilde{Y} \in \mathrm{E}_{f(Z)}(f), U \in \mathbb{D}.$$

The following example illustrates the use of this property.

Consider the problem

$$\max_{x\in\mathbb{D}} \|CX\|_F^2,$$

$$\mathbb{D} = \{X \in \mathbb{R}^{n\times n} | \underline{X} \leqslant X \leqslant \overline{X}, X \geqslant 0\},$$

where

$$C = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}, \qquad \underline{X} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \qquad \overline{X} = \begin{pmatrix} 4 & 5 \\ 3 & 6 \end{pmatrix}.$$

We can evaluate the gradient of f as:

$$f'(X) = 2C^T CX.$$

We check whether a point X^0

$$X^0 = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

is a global solution or not. It can be computed that

$$f(X^0) = 95.$$

Take a point $U \in \mathbb{D}$ such that

$$U \geqslant 0 \qquad \text{and} \qquad U = \begin{pmatrix} 3.7 & 4.6 \\ 2.9 & 5.7 \end{pmatrix}.$$

Find a $\widetilde{Y}$ so that $\widetilde{Y} \in \mathrm{E}_{f(X^0)}(f)$ and

$$\widetilde{Y} = \begin{pmatrix} 0.027273 & 0.036364 \\ 0.009091 & 0.027273 \end{pmatrix}.$$

If we evaluate $\langle f'(\widetilde{Y}), U - \widetilde{Y} \rangle_F$, then we have $\langle f'(\widetilde{Y}), U - \widetilde{Y} \rangle_F = 8.018182 > 0$ which means that X^0 is not a global solution.In fact, the global solution X^* is:

$$X^* = \begin{pmatrix} 4 & 5 \\ 3 & 6 \end{pmatrix}.$$

3 Approximation of the Level Set and Algorithm

As we have seen in Section 2 that in order to check condition (2.5), we need to solve the following semidefinite linear programming for each given $Y \in \mathrm{E}_{f(Z)}(f)$:

$$\max_{X \in \mathbb{D}} \langle f'(Y), X \rangle. \tag{3.1}$$

For this purpose, we need to approximate the level set of the function f with a finite number of points so that one could solve a finite number of problems (3.1).

Definition 3.1. *The set A_Z^m defined for a given $m \in \mathbb{N}$ and $Z \in \mathbb{R}^{n \times n}$ by*

$$A_Z^m = \{Y^1, Y^2, ..., Y^m | Y^i \in \mathrm{E}_{f(Z)}(f), i = 1, 2, ..., m\} \qquad (3.2).$$

is called an approximation set to the level set $\mathrm{E}_{f(Z)}(f)$ at the point Z.

Assume that A_Z^m is given and $\mathbb{D}$ is compact in $\mathbb{R}^{n \times n}$. Let $U^i, i = 1, 2, ..., m$ be the solutions to the following problems:

$$\langle f'(Y^i), U^i \rangle_F = \max_{X \in \mathbb{D}} \langle f'(Y^i), X \rangle_F. \tag{3.3}$$

Define θ_m as follows:

$$\theta_m = \max_{i=1,2..,m} \langle f'(Y^i), X \rangle_F.$$

Lemma 3.1. *If there is a point $Y^i \in A_Z^m$ for $Z \in \mathbb{D}$ such that $\langle f'(Y^i), U^i - Y^i \rangle_F > 0$, where U^i satisfies (1.2),then*

$$f(U^i) > f(Z).$$

Proof. By the definition of U^i, we have

$$\langle f'(Y^i), U^i - Y^i \rangle = \max_{X \in \mathbb{D}} \langle f'(Y^i), X - Y^i \rangle.$$

Since f is convex, then we have

$$f(U) - f(V) \geqslant \langle f'(V), U - V \rangle_F$$

for all $U, V \in \mathbb{R}^{n\times n}$. Therefore, the assumption in the lemma implies that

$$f(U^i) - f(Z) = f(U^i) - f(Y^i) \geqslant \langle f'(Y^i), U^i - Y^i \rangle_F > 0.$$

The proof is complete.

Now we can formulate an algorithm for finding an approximate solutions for problem (2.4) **Algorithm (CDCP)**

Input: A convex differentiable function f and a compact set $\mathbb{D}$ in $\mathbb{R}^{n\times n}$.

Output: An approximate solution X to (2.4).

Step 1. Choose a point $X^k \in \mathbb{D}$. Set $k := 0$.

Step 2. Find a local maximizer $Z^k \in \mathbb{D}$ of problem (2.4)using one of existing methods of semidefinite nonconvex programming starting with X^k.

Step 3. Construct an approximation set $A^m_{Z^k}$ at the point Z^k.

Step 4. For each $Y^i \in A^m_{Z^k}$ solve semidefinite linear programming

$$\max_{X\in\mathbb{D}} \langle f'(Y^i), X \rangle_F.$$

Let $U^i, i = 1, 2, ..., m$ be solutions,i.e.,

$$\langle f'(Y^i), U^i \rangle_F = \max_{X\in\mathbb{D}} \langle f'(Y^i), X \rangle_F.$$

Step 5. Find a number $j \in \{1, 2, ..., m\}$ such that

$$\theta_m = \langle f'(Y^j), U^j - Y^j \rangle_F = \max_{i=1,2..,m} \langle f'(Y^i), U^i - Y^i \rangle_F$$

Step 6. If $\theta^k_m \leqslant 0$ then terminate and Z^k is an approximate solution.

Step 7. Set $X^{k+1} := U^j, k := k + 1$ and go to step 2.

We note that the algorithm generates a sequence of local maximizers $\{Z^k\}$ of problem (2.4) such that

$$f(Z^{k+1}) \geqslant f(Z^k), k = 0, 1...$$

This gives us an opportunity to find an approximate global solution in (2.4) using standard approach of semidefinite programming.

4 Conclusion

For the first time, we introduced so-called semidefinite concave programming or equivalently semidefinite convex maximization problem. Unlike semidefinite convex programming, the problem is nonconvex and NP hard. We derived global optimality conditions by extending a result of Strekalovsky [1] for semidefinite programming. Based on the global optimality conditions, we propose an algorithm for solving the problem. Subproblems of the algorithm are semidefinite linear programming.

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