On a mixed means inequality

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Abstract

In this paper, we prove the conjecture of K. Kedlaya [Amer. Math. Monthly., **106**(1999), 355–358].

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1 Introduction

In [1], Kedlaya conjectured following inequality:

$$s_n\left(L_n^k - R_n^k\right) \geq s_{n-1}\left(L_{n-1}^k - R_{n-1}^k\right) \qquad k = \{r, s\} \tag{1}$$

where

$$L_n^k = \left(\frac{1}{s_n}\sum_{i=1}^{n} w_i \left[\frac{1}{s_i}\sum_{k=1}^{i} w_k a_k^s\right]^{\frac{r}{s}}\right)^{\frac{k}{r}}$$

and

$$R_n^k = \left(\frac{1}{s_n}\sum_{i=1}^{n} w_i \left[\frac{1}{s_i}\sum_{k=1}^{i} w_k a_k^r\right]^{\frac{s}{r}}\right)^{\frac{k}{s}}.$$

Lemma 1.1 (Jensen's Inequality)**.** *If $x_i \geq 0$, $y_i > 0$ and f is a convex function on $[0, +\infty)$ then*

$$\sum y_i f\left(\frac{x_i}{y_i}\right) \geq \left(\sum y_i\right) f\left(\frac{\sum x_i}{\sum y_i}\right). \tag{2}$$

2 Main Results

Theorem 2.1. *If $s \geq r$ and $s_n w_k - s_k w_n \geq 0$ for $1 \leq k \leq n-1$ then*

$$s_n(L_n^k - R_n^k) \geq s_{n-1}(L_{n-1}^k - R_{n-1}^k) \tag{3}$$

is valid for

i) if $k = s > 0$. In case $s < 0$, (3) reversed,

ii) if $k = r > 0$ then (3) holds. In case $r < 0$ then (3) is reversed.

Proof. In case $k = s > 0$, $r \neq 0$, our inequality is equivalent to

$$s_n \geq s_{n-1} \left[\frac{s_n \sum_{i=1}^{n-1} w_i (M_i^{[s]}(a;w))^r}{s_{n-1} \sum_{i=1}^{n} w_i (M_i^{[s]}(a;w))^r} \right]^{\frac{s}{r}} + w_n \left[\frac{\sum_{i=1}^{n} w_i a_i^r}{\sum_{i=1}^{n} w_i (M_i^{[s]}(a;w))^r} \right]^{\frac{s}{r}}. \tag{4}$$

By choosing

$$\begin{array}{ll}
x_1 = (s_n w_1 - s_1 w_n) \cdot (M_1^{[s]}(x))^r, & y_1 = (s_n w_1 - s_1 w_n) \cdot (M_1^{[s]}(x))^r, \\
x_2 = (s_n w_2 - s_2 w_n) \cdot (M_2^{[s]}(x))^r, & y_2 = (s_n w_2 - s_2 w_n) \cdot (M_2^{[s]}(x))^r, \\
\cdots & \cdots \\
x_{n-1} = (s_n w_{n-1} - s_{n-1} w_n) \cdot (M_{n-1}^{[s]}(x))^r, & y_{n-1} = (s_n w_{n-1} - s_{n-1} w_n) \cdot (M_{n-1}^{[s]}(x))^r, \\
x_n = s_1 w_n \cdot (M_1^{[s]}(x))^r, & y_n = s_1 w_n \cdot (M_2^{[s]}(x))^r, \\
x_{n+1} = s_2 w_n \cdot (M_2^{[s]}(x))^r, & y_{n+1} = s_2 w_n \cdot (M_3^{[s]}(x))^r, \\
\cdots & \cdots \\
x_{2n-2} = s_{n-1} w_n \cdot (M_{n-1}^{[s]}(x))^r, & y_{2n-2} = s_{n-1} w_n \cdot (M_n^{[s]}(x))^r
\end{array}$$

and $f(x) = x^{\frac{s}{r}}$, after that using Jensen's Inequality (2) we have

$$\begin{aligned}
&\left[\frac{s_n \sum_{i=1}^{n-1} w_i (M_i^{[s]}(a;w))^r}{s_{n-1} \sum_{i=1}^{n} w_i (M_i^{[s]}(a;w))^r} \right]^{\frac{s}{r}} \\
&\qquad = \left[\frac{\sum_{i=1}^{n-1} (s_n w_i - s_i w_n)(M_i^{[s]}(a;w))^r + s_i w_n (M_i^{[s]}(a;w))^r}{\sum_{i=1}^{n-1} (s_n w_i - s_i w_n)(M_i^{[s]}(a;w))^r + s_i w_n (M_{i+1}^{[s]}(a;w))^r} \right]^{\frac{s}{r}} \qquad (5) \\
&\qquad \leq \frac{\sum_{i=1}^{n-1} (s_n w_i - s_i w_n)(M_i^{[s]}(a;w))^r + s_i w_n \frac{(M_i^{[s]}(a;w))^s}{(M_{i+1}^{[s]}(a;w))^{s-r}}}{s_{n-1} \sum_{i=1}^{n} w_i (M_i^{[s]}(a;w))^r}.
\end{aligned}$$

And again by Jensen's Inequality we get

$$\left[\frac{\sum_{i=1}^{n} w_i a_i^r}{\sum_{i=1}^{n} w_i (M_i^{[s]}(a;w))^r} \right]^{\frac{s}{r}} \leq \frac{\sum_{i=1}^{n} w_i \frac{a_i^s}{(M_i^{[s]}(a;w))^{s-r}}}{\sum_{i=1}^{n} w_i (M_i^{[s]}(a;w))^r}. \tag{6}$$

By multiplying (5) by s_{n-1}, (6) by w_n and adding we get our result. The case when $r = 0$, substituting $x_i \to x_i^s$ in Kedlaya's inequality, we get out (see [1]). The case when $s = 0$, it is obvious. The case $r < s < 0$, it is reversed by Jensen's inequality. In case $k = r$, we can prove similar fashion. $\square$

Theorem 2.2. *Define*

$$L(k)=\begin{cases} s_k\prod\limits_{i=1}^{k}A_i(a;w)^{\frac{w_i}{s_k}}+\sum\limits_{i=k+1}^{n}w_iG_i(a;w), & k\geq 1, k\in\mathbb{N},\\ \sum\limits_{i=1}^{n}w_iG_i(a;w), & k=0\end{cases}.$$

If $\alpha_k=s_ks_{k-1}-w_k\sum\limits_{i=1}^{k-1}s_i\geq 0$ *then*

$$\forall k\in\{1,2,...,n\}:\ L(k)\geq L(k-1). \tag{7}$$

Proof. Using Weighted Arithmetic-Geometric Inequality, we get

$$\begin{aligned} L(k-1) &= s_{k-1}\prod_{i=1}^{k-1}A_i(a;w)^{\frac{w_i}{s_{k-1}}}+w_kG_k(a;w)+\sum_{i=k+1}^{n}w_iG_i(a;w)\\ &= \left(s_{k-1}\left[1^{\alpha_k}\prod_{i=1}^{k-1}\left(\frac{A_i(a;w)}{A_{i+1}(a;w)}\right)^{w_ks_i}\right]^{\frac{1}{s_ks_{k-1}}}+w_k\left[\prod_{i=1}^{k}\left(\frac{a_i}{A_i(a;w)}\right)^{w_k}\right]^{\frac{1}{s_k}}\right)\\ &\quad\times\prod_{i=1}^{k}A_i(a;w)^{\frac{w_i}{s_k}}+\sum_{i=k+1}^{n}w_iG_i(a;w)\\ &\leq \left(\frac{1}{s_k}\left[\sum_{i=1}^{k-1}w_ks_i\frac{A_i(a;w)}{A_{i+1}(a;w)}+\alpha_k\right]+\frac{w_k}{s_k}\left[\sum_{i=1}^{k}\frac{w_ia_i}{A_i(a;w)}\right]\right)\\ &\quad\times\prod_{i=1}^{k}A_i(a;w)^{\frac{w_i}{s_k}}+\sum_{i=k+1}^{n}w_iG_i(a;w)\\ &= s_k\prod_{i=1}^{k}A_i(a;w)^{\frac{w_i}{s_k}}+\sum_{i=k+1}^{n}w_iG_i(a;w)\\ &= L(k). \end{aligned}$$

□

Theorem 2.3. *Define*

$$N(k)=\begin{cases} s_k\left[\sum\limits_{i=1}^{k}\frac{w_i(M_i^{[s]}(a;w))^r}{s_k}\right]^{\frac{s}{r}}+\sum\limits_{i=k+1}^{n}w_i(M_i^{[r]}(a;w))^s, & k\geq 1, k\in\mathbb{N},\\ \sum\limits_{i=1}^{n}w_i(M_i^{[r]}(a))^s, & k=0\end{cases}.$$

and

$$M(k)=\begin{cases} s_k\left[\sum\limits_{i=1}^{k}\frac{w_i(M_i^{[r]}(a;w))^s}{s_k}\right]^{\frac{r}{s}}+\sum\limits_{i=k+1}^{n}w_i(M_i^{[s]}(a;w))^r, & k\geq 1, k\in\mathbb{N},\\ \sum\limits_{i=1}^{n}w_i(M_i^{[s]}(a))^r, & k=0\end{cases}.$$

If $s \geq r$ *and* $s_k w_i - s_i w_k \geq 0$ *for* $1 \leq i \leq k-1, k = 1, ..., n$ *then*

$$\forall k \in \{1, 2, ..., n\} : \ N(k-1) \leq N(k) \tag{8}$$

is valid for if $s > 0$*. In case* $s < 0$*, (8) reversed.*

$$M(k-1) \leq M(k) \tag{9}$$

is valid for $r > 0$*. In case* $r < 0$*, (9) reversed.*

Proof. In case $s > 0$, $r \neq 0$, using Jensen's Inequality, we get

$$\begin{aligned}
N(k-1) &= s_{k-1}\left[\sum_{i=1}^{k-1}\frac{w_i(M_i^{[s]}(a;w))^r}{s_{k-1}}\right]^{\frac{s}{r}} + w_k(M_k^{[r]}(a))^s + \sum_{i=k+1}^{n} w_i(M_i^{[r]}(a;w))^s \\
&= \left(s_{k-1}\left[\frac{s_k\sum\limits_{i=1}^{k-1} w_i(M_i^{[s]}(a))^r}{s_{k-1}\sum\limits_{i=1}^{k} w_i(M_i^{[s]}(a;w))^r}\right]^{\frac{s}{r}} + w_k\left[\frac{\sum\limits_{i=1}^{k} w_i a_i^r}{\sum\limits_{i=1}^{k} w_i(M_i^{[r]}(a))^r}\right]^{\frac{s}{r}}\right) \\
&\quad \times \left[\sum_{i=1}^{k}\frac{w_i(M_i^{[s]}(a;w))^r}{s_k}\right]^{\frac{s}{r}} + \sum_{i=k+1}^{n} w_i(M_i^{[r]}(a;w))^s \\
&= \left(s_{k-1}\left[\frac{\sum\limits_{i=1}^{k-1}(s_k w_i - s_i w_k)(M_i^{[s]}(a))^r + s_i w_k(M_i^{[s]}(a))^r}{\sum\limits_{i=1}^{k}(s_k w_i - s_i w_k)(M_i^{[s]}(a;w))^r + s_i w_k(M_{i+1}^{[s]}(a;w))^r}\right]^{\frac{s}{r}}\right. \\
&\quad \left. + w_k\left[\frac{\sum\limits_{i=1}^{k} w_i a_i^r}{\sum\limits_{i=1}^{k} w_i(M_i^{[r]}(a))^r}\right]^{\frac{s}{r}}\right) \times \left[\sum_{i=1}^{k}\frac{w_i(M_i^{[s]}(a;w))^r}{s_k}\right]^{\frac{s}{r}} \\
&\quad + \sum_{i=k+1}^{n} w_i(M_i^{[r]}(a;w))^s \\
&= s_k\left[\sum_{i=1}^{k}\frac{w_i(M_i^{[s]}(a;w))^r}{s_k}\right]^{\frac{s}{r}} + \sum_{i=k+1}^{n} w_i(M_i^{[r]}(a;w))^s \\
&\leq N(k).
\end{aligned}$$

Last inequality is reversed for $s < 0$ by Jensen's Inequality. The case when $r = 0$, substituting $x_i \to x_i^s$ then it is implies (7). And it is obvious, when $s = 0$. Similarly we can prove that $M(k-1) \leq M(k)$. □

Remark 1. *From (3), (8) and (9) we can get Tarnavas's result (see [3]). And (7) is sharpening of Holland's result ([2]).*

References

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