# Derivative Variation Approach for Some Class of Optimal Control

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**Abstract:** In this paper, the authors introduce so-called derivative variation approach to some class of optimal control problems. Based on derivative variation, we derive new optimality conditions for the original problem in class of continuously differentiable control functions. An algorithm has been constructed for solving the problem using the optimality conditions. Some numerical examples are given.
**Keywords:** optimal control, derivative variation, algorithm

## 1 Introduction

Theory of optimal control problems has been developed in [1]. Optimal control problem has many applications in science and technology. Most of numerical methods and algorithms of optimal control have been based on Maximum Principle. The class of admissible controls is usually defined in class of piecewise continuous or measurable vector-functions. On the other hand, many engineering problems such as robust design and optimal control of inertial systems with safety requirements require continuous control variables [2] and [3]. In order to solve such problems, interior variation approaches have been developed in [7] and [8].
Some difficulties faced with numerical computations of smooth control and practical importance of such computations have been reported in [5] and [4] by Teo et al. Moreover, the authors in their work had developed a numerical algorithm which computes almost smooth controls.
In this paper, we propose another approach for solving optimal control problem in class of continuously differentiable control functions by introducing so-called derivative variations. Derivative variation allows us to formulate necessary conditions for the original problem. We illustrate the approach with illustrative examples.

## 2 Optimal Control and Derivative Variation

Consider the problem.

$$\begin{aligned}
&J(u) = \varphi(x(t_1)) + \int_{t_0}^{t_1} F(x,u,t)dt \to min, \\
&\dot{x} = f(x,u,t), \\
&x(t_0) = x^0,
\end{aligned} \tag{1}$$

where $t \in T = [t_0, t_1]$, $x(t) = (x_1(t), .., x_n(t))^T, u(t) = (u_1(t), .., u_r(t))^T$ $\varphi :\ R^n \to R$ is a differentiable function. The functional $F : R^n \times R^n \times R \to R$ is continuous in all arguments and continuously differentiable in $x$ and $u$. Here $x^0 \in R^n$ is an initial state, $t_0, t_1$ and $x^0$ are given. The vector function $f = (f_1, .., f_n)$ is continuous in its arguments $(x, u, t)$ together with its partial derivatives with respect to $x$ and satisfies Lipshitz's condition in $x$ with the same constant $L$ for all $u \in R^r,\ t \in T$.

$$\|f(x + \Delta x, u, t) - f(x, u, t)\| \leq L\|x\|, \tag{2}$$

Also, assume that $u(t) \in C^1_r([t_0, t_1]),\ u(t) \not\equiv const,\ \|\dot{u}(t)\| \leq M < \infty,\ t \in T.$ Introduce the set as:

$$S = \{\delta \in C^1([t_0, t_1])|\ \|\delta(t)\| \leq K, \delta(t_0) = \delta(t_1) = 0\}.$$

Introduce derivative variations of $u(t) \in C^1_r$ as:

$$\widetilde{u}_\varepsilon = u(t) + \varepsilon\delta\dot{u}(t), \tag{3}$$

for all $\delta \in S$, $\varepsilon \in R$ and $t \in T$. We define a weak optimal control:

**Definition 1** *An admissible control $u^*$ is called a weak optimal control to problem* (1) *if there exist a positive number $\beta > 0$ such that*

$$J(u^*) \leq J(u)$$

*holds for all $u \in C^1_r(T)$ satisfying*

$$\|u - u^*\|_{c^1} < \beta.$$

Let a process $(x^*, u^*)$ be a weak optimal process. Define by $\delta_\varepsilon u(t)$ an increment of $u$ at $t$.

$$\Delta_\varepsilon u(t) = \widetilde{u}_\varepsilon(t) - u(t) = \varepsilon\dot{u}(t)\delta(t); \tag{4}$$

$$\begin{cases} \dot{\psi}(t) = -\frac{\partial H(\varphi,x,u,t)}{\partial x} \\ \psi(t_1) = -\frac{\partial \varphi(x(t_1))}{\partial x} \end{cases} \tag{5}$$

$$\Delta_{\widetilde{u}_\varepsilon} H(\psi, x, u, t) = H(\psi, x, \widetilde{u}_\varepsilon, t) - H(\psi, x, u, t),$$

$$\Delta_\varepsilon J(u) = J(\tilde{u}_\varepsilon) - J(u) = -\int\limits_{t_0}^{t_1} \Delta_{\tilde{u}_\varepsilon} H(\psi, x, u, t)dt + \eta_{\tilde{u}_\varepsilon},$$

$$\eta_{\widetilde{u}} = O_\varphi(\|\Delta_\varepsilon x(t_1)\|) - \int\limits_{t_0}^{t_1} O_H(\|\Delta_\varepsilon x(t)\|)dt - \int\limits_{t_0}^{t_1} \left\langle \Delta_{\widetilde{u}_\varepsilon} \frac{\partial H}{\partial x}, \Delta_\varepsilon x(t) \right\rangle dt.$$

On the other hand, since $H(\cdot)$ is differentiable with respect to $u$ then there is an estimate by Vasilieva [7] for $\Delta_\varepsilon x(t)$ as

$$\|\Delta_\varepsilon x(t)\| \leq L \int\limits_{t_0}^{t_1} \|\Delta_\varepsilon u(t)\| dt. \tag{6}$$

Obviously, in our case formula (6) provides the following estimate:

$$\|\Delta_\varepsilon x(t)\| \leq L \int\limits_{t_0}^{t_1} \varepsilon \dot{u}(t)\delta(t)dt \leq L \int\limits_{t_0}^{t_1} |\varepsilon||\dot{u}(t)||\delta(t)|dt \leq L|\varepsilon|MK(t_1 - t). \tag{7}$$

Thus we have

$$O_\varphi(\|\Delta_\varepsilon x(t_1)\|) \sim O(\varepsilon) \text{ and } \int\limits_{t_0}^{t_1} O_H(\|\Delta_\varepsilon x(t_1)\|)dt \sim O(\varepsilon) \tag{8}$$

We can write the following:

$$\Delta_{\widetilde{u}_\varepsilon} H(\psi, x, u, t) = \left\langle \frac{\partial H(\psi, x, u, t)}{\partial u}, \Delta_\varepsilon u(t) \right\rangle + O_H(\|\Delta_\varepsilon u(t)\|). \tag{9}$$

Taking into account the Lipschitz condition (2) and (8) and substituting (3) in formulas (6)-(8), we obtain

$$\eta_{\widetilde{u}_\varepsilon} \sim O(\varepsilon). \tag{10}$$

Then formula $\Delta_\varepsilon J(u)$ reduces to

$$\Delta_\varepsilon J(u) = -\varepsilon \int\limits_{t_0}^{t_1} \left\langle \frac{\partial H(\psi, x, u, t)}{\partial u}, \dot{u} \right\rangle \delta(t)dt + O(\varepsilon), \tag{11}$$

$$\lim_{\varepsilon \to 0} \frac{O(\varepsilon)}{\varepsilon} = 0.$$

Now we are ready to formulate necessary conditions for problem (1) in class of continuously differentiable functions with bounded derivatives based on the derivative variation.

**Theorem 1** *Assume that an admissible process* $(x^*, u^*)$ *is weak optimal to problem* (1)*. Then the following conditions*

$$\left\langle \frac{\partial H(\psi^*, x^*, u^*, t)}{\partial u}, \dot{u}^*(t) \right\rangle = 0$$

*are satisfied for all* $t \in T$*, where* $\psi^* = \psi^*(t)$ *is the solution to system* (5) *for* $u = u^*(t)$ *and* $x = x^*(t)$*.*

**Proof.** Let $\{x^*, u^*\}$ be a weak optimal process. Then formula (11) at the optimal control has the form:

$$\Delta_\varepsilon J(u^*) = J(u_\varepsilon(t)) - J(u^*) = -\varepsilon \int_{t_0}^{t_1} \left\langle \frac{\partial H(\psi^*, x^*, u^*, t)}{\partial u}, \dot{u}^* \right\rangle \delta(t) dt + O(\varepsilon), \quad (12)$$

where $\delta(t) \in S$ and $\frac{O(\varepsilon)}{\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} 0, \; t \in [t_0, t_1]$. Moreover, $\|O(\varepsilon)\| \leq Q \|\varepsilon^2\|, \; Q = const > 0$.

Denote by $P$ the main term in (12). That is

$$P = \int_{t_0}^{t_1} \left\langle \frac{\partial H(\psi^*, x^*, u^*, t)}{\partial u}, \dot{u}^* \right\rangle \delta(t) dt.$$

Then we have

$$\Delta_\varepsilon J(u^*) = -\varepsilon P + O(\varepsilon),$$

where $\lim\limits_{\varepsilon \to 0} \frac{O(\varepsilon)}{\varepsilon} = 0$. To show $P = 0$, assume the contrary, that is $P \neq 0$. Now we can write $\Delta_\varepsilon J(u)$ as

$$\Delta_\varepsilon J(u^*) = -\varepsilon \left[ P + \frac{O(\varepsilon)}{\varepsilon} \right].$$

For sufficiently small $\varepsilon$, the sign of $\Delta_\varepsilon J(u^*)$ is determined by the sign of $\varepsilon P$. By taking $\varepsilon$ as $\varepsilon = +\varepsilon' sign(P)$ for small $\varepsilon' > 0$, we have $\Delta_\varepsilon J(u^*) = J(u_\varepsilon(t)) - J(u^*) < 0$ which contradicts the definition of the weak control. Consequently, $P = 0$ or equivalently

$$\int_{t_0}^{t_1} \left\langle \frac{\partial H(\psi^*, x^*, u^*, t)}{\partial u}, \dot{u}^* \right\rangle \delta(t) dt = 0,$$

for all $\delta(t) \in S$. If we apply the lemma ([8],p.65) to the above equality, we obtain

$$\left\langle \frac{\partial H(\psi^*, x^*, u^*, t)}{\partial u}, \dot{u}^* \right\rangle = 0, \quad t \in T$$

which completes the proof. ■

# 3 Derivative Variation Based Algorithm

For the purpose of constructing an algorithm for solving problem (1), we additionaly assume that $u(t) \in C_r^\infty(T)$ and rewrite the increment formula $\Delta_\varepsilon J(u)$ as:

$$J(u_\varepsilon) - J(u) = -\varepsilon P(u) + O(\varepsilon),$$

where $P(u) = \int_{t_0}^{t_1} \left\langle \frac{\partial H(\psi,x,u,t)}{\partial u}, \dot{u} \right\rangle \delta(t) dt$.

Introduce the function $E(u,t)$ at $(u,t)$ as:

$$E(u,t) = \left\langle \frac{\partial H(\psi, x, u, t)}{\partial u}, \dot{u} \right\rangle. \tag{13}$$

Denote by $C$ the maximum value of $E(t,u)$ on $T$, i.e,

$$C = \max_{t \in T} \|E(u,t)\|.$$

Since parameters $\varepsilon$ and $\delta(t)$ vary arbitrarily, we can specify these parameters as follows:

a) $\varepsilon \in (0, \varepsilon_0), \varepsilon_0 > 0$ is a given small number

b)

$$\delta(t) = \frac{(t - t_0)(t_1 - t)}{(t_1 - t_0)} E(u,t), \forall u \in R^r. \tag{14}$$

It is clear that $\|\delta(t)\| \leq t_1 - t_0$. Then the increment formula is:

$$J(u_\varepsilon) - J(u) = -\varepsilon \int_{t_0}^{t_1} \frac{(t - t_0)(t_1 - t)}{(t_1 - t_0)C} E^2(u,t) dt + O(\varepsilon), \ \varepsilon \in (0, \varepsilon_0).$$

From here, we conclude that there exists $\varepsilon^*$ such that

$$J(u_\varepsilon) \leq J(u), \quad \forall \varepsilon \in (0, \varepsilon^*). \tag{15}$$

Clearly, if $u^*$ is weak optimal then $E(u^*, t) = 0, \forall t \in T$.

Now we are ready to construct an algorithm for problem (1).

**Algorithm OCP**

**Step 1.** Let $k := 0$ and $u^k \in C_r^\infty$ be an admissible control and

$$x^k = x^k(u^k, t),\ \psi^k = \psi^k(x^k, u^k, t)$$

are solutions of systems (5) for $u = u^k$ and $x = x^k$.

**Step 2.** Compute $E(u^k, t)$ by formula (13), i.e

$$E(u^k, t) = \left\langle \frac{\partial H(\psi^k, u^k, t)}{\partial u}, \dot{u}^k \right\rangle,\ \forall t \in T$$

where $H(\psi^k, u^k, t) = \langle \psi^k(t), f(x^k, u^k, t) \rangle - F(x^k, u^k, t)$.

**Step 3.** If $E(u^k, t) = 0, \forall t \in T$ then stop $u^* = u^k$ is a weak optimal control.

**Step 4.** Choose $\delta_k(t) \in S$ such that

$$\delta_k(t) = \frac{(t - t_0)(t_1 - t)}{(t_1 - t_0)C_k} E(u^k, t),$$

where $C_k = \max_{t \in T} E(u^k, t)$.

**Step 5.** Construct $u^{k+1}$ as

$$u^{k+1} = u^k + \varepsilon_k \delta_k \dot{u}^k, \quad \varepsilon_k > 0,$$

where $\varepsilon_k$ satisfies the condition

$$J(u^k + \varepsilon \delta_k \dot{u}^k) < J(u^k), \quad \varepsilon > 0.$$

**Step 6.** Set $k := k + 1$ and go to Step 1.

Denote by $P_k$ the following integral

$$P_k = P(u^k) = \int_{t_0}^{t_1} E(u^k, t)\delta_k(t)dt$$

**Theorem 2** *Assume that* $\inf_{u \in C_r^\infty} J(u) > -\infty$, *and conditions* (2) *and* (14) *hold. Then for a sequence* $\{u^k\}$ *generated by Algorithm OCP the condition*

$$\lim_{k \to \infty} P_k = 0$$

*is satisfied.*

**Proof.** By construction of $\{u^k\}$, we have

$$J(u^k_\varepsilon) - J(u^k) = -\varepsilon P(u^k) + O(\varepsilon).$$

Taking into account of (2) and (12), we obtain

$$J(u^{k+1}) - J(u^k) \leq -\varepsilon P_k + Q|\varepsilon|^2, \forall \varepsilon > 0,$$

$$J(u^k) - J(u^{k+1}) \geq \varepsilon P_k - Q\varepsilon^2,$$

$$J(u^k) - J(u^{k+1}) \geq \max_{\varepsilon>0}[\varepsilon P_k - Q\varepsilon^2] = \frac{P_k^2}{4Q} > 0.$$

Since $J(\cdot)$ is bounded below, there exist

$$\lim_{k\to\infty} J(u^k) = A > -\infty,$$

$$J(u^k) - J(u^{k+1}) > \frac{P_k^2}{4Q} > 0.$$

Hence, we obtain $\lim_{k\to\infty} P_k = 0$ which completes the proof. ■

**Example 1.**

$$\begin{aligned} &J(u) = x(1) \to \min, \\ &\dot{x} = x + u^2(t) + u(t), \\ &x(0) = 0, \\ &|u(t)| \leq 1, \quad t \in [0, 1]. \end{aligned}$$

The problem has constant optimal solution $u^* = -0.5$, which generates the optimal trajectory $x^*(t) = \frac{3}{4}e^t - \frac{3}{4}$ for $t \in [0,\ 1]$ and the objective function attains $J^* = J(x^*) \approx 1.2887$.

Let us check control $u^0(t) = t$ for its optimality using Theorem 1. Compose the Hamiltonian function as:

$$H = \Psi(t)(x + u^2 + u),$$

The conjugate system is:

$$\begin{cases} \dot{\Psi} = -\Psi \\ \Psi(1) = -1. \end{cases}$$

$\Psi(t) = -e^{-(t-1)}$ is the solution of the system.

$$\frac{\partial H^0[t]}{\partial u} = 2 \cdot u^0 \cdot \Psi + \Psi = \Psi \cdot [2u^0 + 1]$$

$$\frac{\partial H^0[t]}{\partial u} = -e^{-(t-1)} \cdot [2t + 1], \quad t \in [0, 1]$$

Since $\dot{u}(t) = 1$ and

$$\left\langle \frac{\partial H^0[t]}{\partial u}, \dot{u}(t) \right\rangle = -e^{-(t-1)} \cdot [2t+1] < 0, \quad t \in [0,1]$$

$u^0(t)$ is not weak control. In order to improve $u^0(t)$, we construct a new control via derivative variational approach

$$u_\varepsilon(t) = u^0(t) + \varepsilon \cdot \delta(t) \cdot \dot{u}(t), \quad t \in [0,1].$$

We set $\delta(t) = t(t-1), \quad \varepsilon = 1.$

$$u_\varepsilon(t) = t + t \cdot (t-1) = t^2, \quad u^1(t) = t^2.$$

It can be easily checked that

$$J(u^0) > J(u^1).$$

Now we check the optimality conditions of Theorem 1 at $u^1(t)$:

$$H^1[t] = \Psi \cdot (x + u^2 + u), \quad \dot{u}^1(t) = 2 \cdot t,$$
$$\frac{\partial H^1[t]}{\partial u} = \Psi \cdot [2u^1 + 1] = -e^{-(t-1)} \cdot (2t^2 + 1),$$
$$\left\langle \frac{\partial H^1[t]}{\partial u}, \dot{u}^1(t) \right\rangle = -e^{-(t-1)} \cdot (2t^2 + 1) \leq 0, \quad t \in [0,1].$$

It means that $u^1(t)$ is not optimal again. Now we construct a control $u^2(t)$ as follows:

$$u^2(t) = u^1(t) + \varepsilon \cdot \delta(t) \cdot \dot{u}^1(t), \quad \delta(t) = t \cdot (t-1), \quad \varepsilon = 1/2,$$
$$u^2(t) = t^2 + t \cdot (t-1) \cdot \frac{2t}{2} = t^2 + t^3 - t^2 = t^3, \quad \dot{u}^2(t) = 3 \cdot t^2,$$
$$\left\langle \frac{\partial H^2[t]}{\partial u}, \dot{u}^2(t) \right\rangle = -e^{-(t-1)} \cdot (2t^3 + 1) \cdot 3t^2 \leq 0, \quad t \in [0,1].$$

It means that $u^2(t)$ does not satisfy the optimality conditions for a weak optimal control.

Note that the algorithm OCP can be modified easily for problems with control constraints of type $u \in U$. According to (15), we have a flexible choice for parameter $\varepsilon$. We select a parameter $\varepsilon$ from conditions $u_\varepsilon^{k+1} \in U$ and $\varepsilon \in (0, \varepsilon^*)$.

**Example 2.**

$$
\begin{aligned}
\min \quad & J(u) = \int_0^{\pi} x \sin(t) dt - x(\pi) && \text{(16a)} \\
\text{s.t.} \quad & \dot{x} = u, && \text{(16b)} \\
& x(0) = 0, && \text{(16c)} \\
& -1 \le u \le 1, && \text{(16d)} \\
& -1 \le \dot{u} \le 1, && \text{(16e)} \\
& u \in C^1([0,\pi]) && \text{(16f)}
\end{aligned}
$$

We start from describing its classical solution. When only constraints (16b)-(16d) are taken under consideration, an optimal control to the above problem is given by $u^*(t) = sign(t - \frac{\pi}{2})$. In this case the optimal trajectory is

$$
x^*(t) = \begin{cases} -t, & \text{if } 0 \le t \le \frac{\pi}{2}, \\ -\frac{\pi}{2} + t, & \text{if } \frac{\pi}{2} \le t \le \pi \end{cases}
$$

and it attains optimal value $J^* = -\frac{\pi^2}{4} \approx -2.4674$.

Further, we illustrate how the above non-differentiable control can be approximated by smooth one by employing algorithm OCP. In this purpose, we start from initial smooth control $u^0 = -\cos(t)$. Clearly, it holds $x^0(t) = -\sin(t)$ and $J_0 = J(x^0) = -\frac{\pi}{2} \approx -1.5708$. Compose the Hamiltonian function as

$$
H(x, u, p, t) = x \sin(t) + pu.
$$

The conjugate equation is

$$
\begin{cases} \dot{p} = -\sin(t) \\ p(\pi) = -1 \end{cases}
$$

Its solution is $p(t) = \cos(t)$. Furthermore, it holds

$$
\frac{\partial H(t)}{\partial u} = p(t) = \cos(t), \quad t \in [0, \pi].
$$

Since $\dot{u}^0(t) = \sin(t)$ and

$$
\left\langle \frac{\partial H[t]}{\partial u}, \dot{u}(t) \right\rangle = \cos(t)\sin(t) \not\equiv 0, \quad t \in [0, \pi]
$$

$u^0(t)$ is not weak optimal. In order to improve $u^0(t)$, we construct a new control via derivative variational approach

$$
u_\varepsilon(t) = u^0(t) + \varepsilon \cdot \delta(t) \cdot \dot{u}(t), \quad t \in [0, \pi].
$$

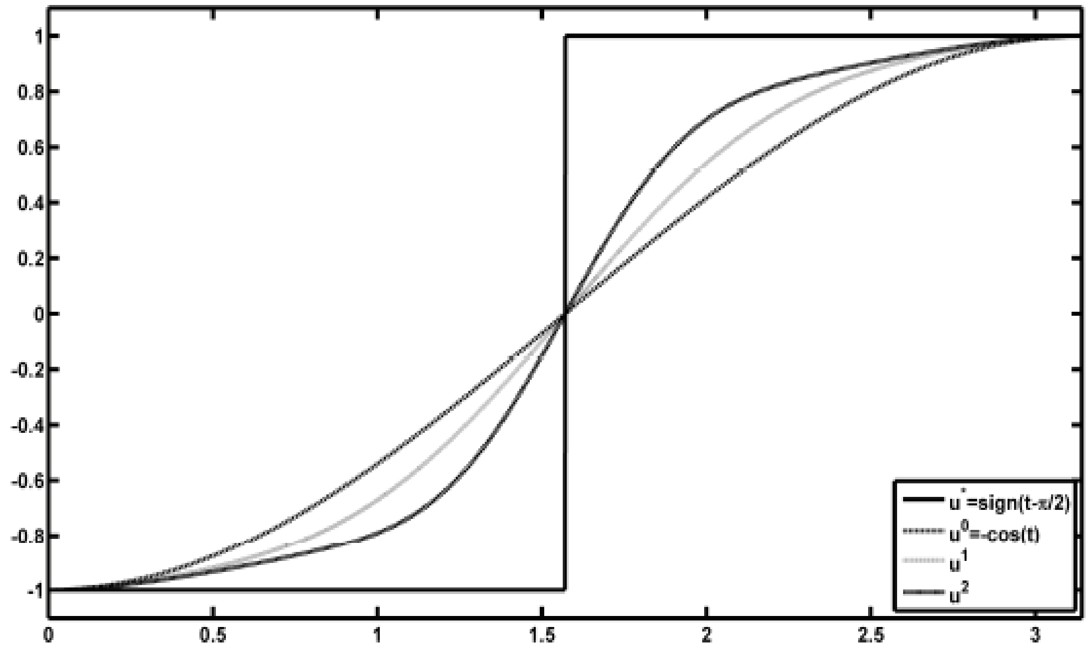


Figure 1: Graphics of controls to Example 2 on $t \in [0, \pi]$.

We set $\delta(t) = \frac{t(\pi-t)}{\pi}\cos(t)\sin(t), \quad \varepsilon = 0.5$. Then it holds

$$u^1(t) = -\cos(t) + \frac{1}{2\pi}t(\pi - t)\cos(t)\sin^2(t).$$

By direct calculation we find

$$x^1(t) = \frac{1}{8\pi}\left[\left(\frac{2}{3}t^2 - \frac{2}{3}\pi t - \frac{52}{27} - 8\pi\right)\sin(t) + \left(\frac{17t}{9} - \frac{8\pi}{9}\right)\cos(t)\right] + \frac{1}{9},$$

and the objective functional attains $J_1 \approx -2.2493$ at $x^1$. In a similar way to the above, we had calculated functions $\dot{u}^1$, $u^2$ and $x^2$, selecting $\varepsilon = 0.5$. Since, closed form description of these functions are too long, we present only calculation results based on them. The optimality conditions are not satisfied at $x^1$ and $x^2$ (Figure 3). $J_2 = J(x^2) \approx -2.2544$. By Theorem 2, it holds $\lim\limits_{k\to+\infty} P(u^k) = 0$.

# Conclusion

We introduced a so-called derivative variation approach to some class of optimal control problems. We formulated an optimality condition for the problem in class of continuously differentiable control functions. Based on the optimality condition,

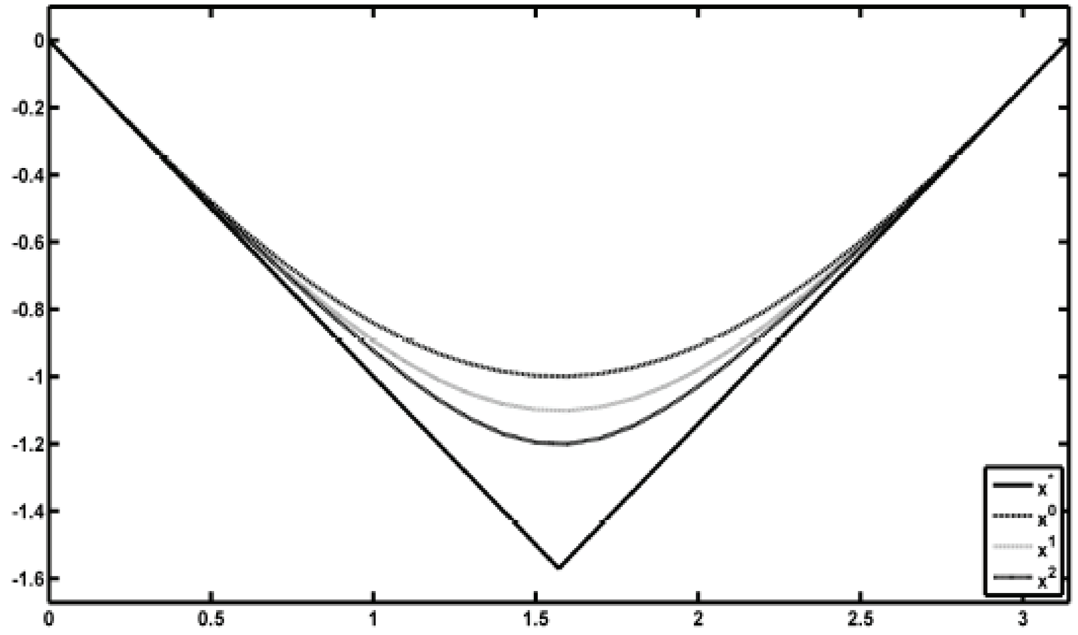


Figure 2: Graphics of processes to Example 2 on $t \in [0, \pi]$.

we constructed an algorithm which converges to a weak solution. Numerical implementations of the algorithm as well as control constrained case will be done in the next paper.

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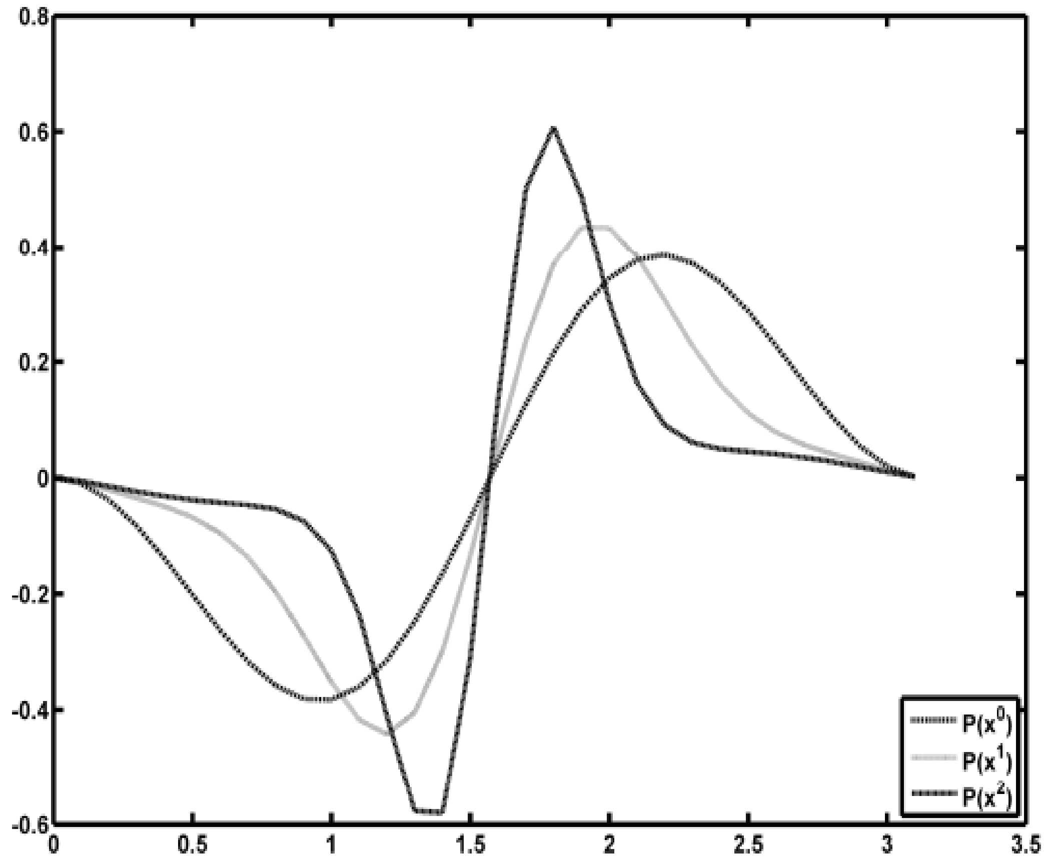


Figure 3: Graphics of optimality conditions on $t \in [0, \pi]$ for Example 2, evaluated at $x^0$, $x^1$ and $x^2$.

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