On the Minimization of Product of Two Concave Functions

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Abstract. We consider a problem of minimizing the product of two concave functions. The problem belongs to a class of global optimization. We derive new global optimality conditions for the problem by applying a result in [1].

1 Introduction

Optimization of a product of convex or concave functions plays important roles not only in optimization theory but also in engineering and economics. A typical example is a bound portfolio optimization problem [6]. The simplest example of multiplicative programming problem is a linear multiplicative programming problem in which a product of two affine functions is minimized over a polytope. The problem is not convex. Therefore, a global solution over a convex feasible set may not be found by standard optimization techniques. In general, multiplicative programming problem can be solved by using an outer approximation algorithm [5].
In the most literature [2–6], so called convex multiplicative programming problem which is a minimization of a product of several convex functions has been studied.

$$\min f(x) = \prod_{j=1}^{p} f_j, \tag{1.1}$$

subject to

$$g_j(x) \leqslant 0, \quad j = 1, 2, ..., m, \tag{1.2}$$

where $f_j : \mathbb{R}^n \to \mathbb{R}, \quad j = 1, 2, ..., p$ and $g_i : \mathbb{R}^n \to \mathbb{R}, \; j = 1, 2, ..., m$ are convex functions.
The following result is well known [4] for solving problem (1.1)-(1.2). First, the following master problem is solved.

$$\min F(x, \lambda) = \sum_{j=1}^{p} \lambda_j f_j, \tag{1.3}$$

subject to

$$g_j(x) \leqslant 0, \quad j = 1, 2, ..., m,$$

$$\prod_{j=1}^{p} \lambda_j \geqslant 1, \quad \lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \geqslant 0. \tag{1.4}$$

Lemma 1.1 [4] Let (x^*, λ^*) be an optimal solution of (1.3)-(1.4). Then x^* is optimal to (1.1)-(1.2).

In this paper we consider a problem of minimizing a product of two concave functions. So far less attention has been paid to this class of problems. Our goal is to fulfill this gap.

2 Quasiconcave functions

Definition 2.1 A function $f : \mathbb{D} \to \mathbb{R}$ is said to be quasiconcave on a convex set $\mathbb{D} \subset R^n$ if

$$f(\alpha x + (1-\alpha)y) \geqslant \min\{f(x), f(y)\}$$

holds for all $x, y \in \mathbb{D}$ and $\alpha \in [0, 1]$.
If f is quasiconcave then $-f$ is called quasiconvex.

Theorem 2.1 A function $f : \mathbb{D} \to \mathbb{R}$ is quasiconvace on $\mathbb{D}$ if and only if the set

$$L_c(f) = \{x \in \mathbb{D} | f(x) \geqslant c\}$$

is convex for all $c \in \mathbb{R}$.
Proof. **Necessity**. Suppose that $c \in \mathbb{R}$ is an arbitrary number and $x, y \in L_c(f)$. By the definition of quasiconcavity, we have

$$f(\alpha x + (1-\alpha)y) \geqslant \min\{f(x), f(y)\} \geqslant c$$

for all $\alpha \in [0, 1]$, which means that the set $L_c(f)$ is convex.
Sufficiency. Let $L_c(f)$ be a convex set for all $c \in \mathbb{R}$. For arbitrary $x, y \in \mathbb{R}^n$, define $c^0 = \min\{f(x), f(y)\}$. Then $x \in L_{c^0}(f)$ and $y \in L_{c^0}(f)$.
Consequently, $\alpha x + (1-\alpha)y \in L_{c^0}(f)$ for any $\alpha \in [0, 1]$. This completes the proof. ■

Consider a problem of minimizing the product of two concave funtions

$$\min_{x \in \mathbb{D}} f = f_1 \cdot f_2 \tag{2.1}$$

where $f_1, f_2 : \mathbb{D} \to \mathbb{R}$ are positive defined concave functions on a convex set $\mathbb{D} \subset \mathbb{R}^n$.

Lemma 2.1 The function f is quasiconcave on $\mathbb{D} \subset \mathbb{R}^n$.
Proof. Introduce the new function $\varphi : \mathbb{D} \to \mathbb{R}$ as $\varphi = \frac{1}{f_2}$.
Obviously, φ is convex and positive on $\mathbb{D}$. Then function f is: $f = \frac{f_1}{\varphi}$.
Define the set $L_c(f)$:

$$L_c(f) = \{x \in |f(x) \geqslant c\}$$

for all positive $c \in R^+$.
We show that $L_c(f)$ is convex. Take points $x, y \in L_c(f)$ and $\alpha \in [0, 1]$.
Then

$$f_1(x) - c\varphi(x) \geqslant 0, \qquad f_1(y) - c\varphi(y) \geqslant 0.$$

Taking into account concavities of functions f_1 and φ, we compute

$$f_1(\alpha x + (1-\alpha)y) - c\varphi(\alpha x + (1-\alpha)y).$$

$f_1(\alpha x + (1-\alpha)y) - c\varphi(\alpha x + (1-\alpha)y) \geqslant \alpha f_1(x) + (1-\alpha)f_1(y) - \alpha c\varphi(x) - (1-\alpha)c\varphi(y) = \alpha[f_1(x) - c\varphi(x)] + (1-\alpha)[f_1(x) - c\varphi(x)] \geqslant 0$
which is equivalent to $f(\alpha x + (1-\alpha)y) \geqslant c$ and means that $\alpha x + (1-\alpha)y \in L_c(f), \quad c > 0$.
Now we are ready to formulate global optimality conditions for problem (2.1).
On the other hand, problem (2.1) can be treated equivalently, as quasiconvex maximization problem

$$\max_{x \in \mathbb{D}} (-f) \tag{2.2}$$

If we apply of Theorem 2.1 [1] to problem (2.2), then we obtain global optimality conditions for the problem in the following proposition.

Theorem 2.2 Let z be a global solution to problem (2.1) and $E_c(f) = \{y \in \mathbb{R}^n | f(y) = c\}$. Then

$$\langle f'(y), x - y \rangle \geqslant 0 \quad \text{for all} \quad y \in E_{f(z)}(f) \quad \text{and} \quad x \in \mathbb{D}. \tag{2.3}$$

If, in addition, $f'(y) \neq 0$ holds for all $y \in E_{f(z)}(f)$, then condition (2.3) is sufficient for $x \in \mathbb{D}$ to be a global solution to problem (2.1)
Proof. Direct application of the result of Theorem 2.1 (pp.19, [1]) follows the proof. ■

It can be computed that

$$f'(x) = f' \cdot f_2 + f \cdot f_2',$$

where

$$f'(x) = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}).$$

In order to conclude that a point $z \in \mathbb{D}$ is not a solution to problem (2.1),Theorem(2.1) tells that we need to find $x, y \in \mathbb{R}^n$ such that

$$\langle f'(y), x - y\rangle < 0, \quad f(y) = f(z), \quad x \in \mathbb{D}.$$

The following example illustrates use of this property.
Example 1

$$\min_{x\in\mathbb{D}} f = (x_1^2 + 4x_1x_2 + 5x_2^2)(2x_1^2 + 2x_1x_2 + 3x_2^2),$$

$$D = \{x \in \mathbb{R}^2 | -2 \leqslant x_1 \leqslant 3, \quad -1 \leqslant x_2 \leqslant 4\}.$$

We can easily evaluate the gradient of f as :

$$f' = [(2x_1 + 4x_2)(2x_1^2 + 2x_1x_2 + 3x_2^2) + (x_1^2 + 4x_1x_2 + 5x_2^2)(4x_1 + 2x_2),$$

$$(4x_1 + 10x_2)(2x_1^2 + 2x_1x_2 + 3x_2^2) + (x_1^2 + 4x_1x_2 + 5x_2^2)(2x_1 + 6x_2)].$$

By Lemma 2.1, f is quasiconcave. Now we want to check whether a point $z = (3, -1)$, which is a local minimizer, is a global solution or not.
Then consider a pair $u = (-1, 1) \in \mathbb{D}$ and $y = (0, \sqrt[4]{2})$ satisfying $f(y) = f(z) = 30$.
Now we have

$$f'(y) = (22\sqrt[4]{8}, 60\sqrt[4]{8}) \text{ and } \langle f'(y), u - y\rangle = -22\sqrt[4]{8} + (60\sqrt[4]{8})(1 - \sqrt[4]{2}) < 0$$

which follows that z is not a global solution. In fact , we can see that the global solution is $z^* = (3, 4)$.

Conclusion

We considered the problem of minimizing a product of two concave functions. The problem is nonconvex. We reduced the problem to quasiconvex maximization problem and then applied a result in [1] obtaining new global optimality conditions. Numerical algorithms and results will be discussed in a next paper.

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