On the Minimization of Product of Two Concave Functions

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Abstract. We consider a problem of minimizing the product of two concave functions. The problem belongs to a class of global optimization. We derive new global optimality conditions for the problem by applying a result in [1].

1 Introduction

Optimization of a product of convex or concave functions plays important roles not only in optimization theory but also in engineering and economics. A typical example is a bound portfolio optimization problem [6]. The simplest example of multiplicative programming problem is a linear multiplicative programming problem in which a product of two affine functions is minimized over a polytope. The problem is not convex. Therefore, a global solution over a convex feasible set may not be found by standard optimization techniques. In general, multiplicative programming problem can be solved by using an outer approximation algorithm [5].

In the most literature [2–6], so called convex multiplicative programming problem which is a minimization of a product of several convex functions has been studied.

\[
\min f(x) = \prod_{j=1}^{p} f_j, \tag{1.1}
\]

subject to

\[
g_j(x) \leq 0, \quad j = 1, 2, \ldots, m, \tag{1.2}
\]

where \( f_j : \mathbb{R}^n \to \mathbb{R}, \quad j = 1, 2, \ldots, p \) and \( g_i : \mathbb{R}^n \to \mathbb{R}, \quad j = 1, 2, \ldots, m \) are convex functions.

The following result is well known [4] for solving problem (1.1)-(1.2). First, the following master problem is solved.

\[
\min F(x, \lambda) = \sum_{j=1}^{p} \lambda_j f_j, \tag{1.3}
\]

subject to

\[
g_j(x) \leq 0, \quad j = 1, 2, \ldots, m,
\]

1
Lemma 1.1 [4] Let \((x^*, \lambda^*)\) be an optimal solution of (1.3)-(1.4). Then \(x^*\) is optimal to (1.1)-(1.2).

In this paper we consider a problem of minimizing a product of two concave functions. So far less attention has been paid to this class of problems. Our goal is to fulfill this gap.

2 Quasiconcave functions

Definition 2.1 A function \(f : \mathbb{D} \rightarrow \mathbb{R}\) is said to be quasiconcave on a convex set \(\mathbb{D} \subset \mathbb{R}^n\) if
\[
f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\}
\]
holds for all \(x, y \in \mathbb{D}\) and \(\alpha \in [0, 1]\).
If \(f\) is quasiconcave then \(-f\) is called quasiconvex.

Theorem 2.1 A function \(f : \mathbb{D} \rightarrow \mathbb{R}\) is quasiconve on \(\mathbb{D}\) if and only if the set
\[
L_c(f) = \{x \in \mathbb{D} | f(x) \geq c\}
\]
is convex for all \(c \in \mathbb{R}\).

Proof. Necessity. Suppose that \(c \in \mathbb{R}\) is an arbitrary number and \(x, y \in L_c(f)\). By the definition of quasiconcavity, we have
\[
f(\alpha x + (1 - \alpha)y) \geq \min\{f(x), f(y)\} \geq c
\]
for all \(\alpha \in [0, 1]\), which means that the set \(L_c(f)\) is convex.

Sufficiency. Let \(L_c(f)\) be a convex set for all \(c \in \mathbb{R}\). For arbitrary \(x, y \in \mathbb{R}^n\), define \(c^0 = \min\{f(x), f(y)\}\). Then \(x \in L_{c^0}(f)\) and \(y \in L_{c^0}(f)\).
Consequently, \(\alpha x + (1 - \alpha)y \in L_{c^0}(f)\) for any \(\alpha \in [0, 1]\). This completes the proof.

Consider a problem of minimizing the product of two concave funtions
\[
\min_{x \in \mathbb{D}} f = f_1 \cdot f_2
\]
where \(f_1, f_2 : \mathbb{D} \rightarrow \mathbb{R}\) are positive defined concave functions on a convex set \(\mathbb{D} \subset \mathbb{R}^n\).
Lemma 2.1 The function \( f \) is quasiconcave on \( D \subset \mathbb{R}^n \).

Proof. Introduce the new function \( \varphi : D \rightarrow \mathbb{R} \) as \( \varphi = \frac{1}{f} \).

Obviously, \( \varphi \) is convex and positive on \( D \). Then function \( f \) is: \( f = \frac{1}{\varphi} \).

Define the set \( L_c(f) \): \[
L_c(f) = \{ x \in \mathbb{R} | f(x) \geq c \}
\]
for all positive \( c \in \mathbb{R}^+ \).

We show that \( L_c(f) \) is convex. Take points \( x, y \in L_c(f) \) and \( \alpha \in [0, 1] \).

Then \[
f_1(x) - c\varphi(x) \geq 0, \quad f_1(y) - c\varphi(y) \geq 0.
\]

Taking into account concavities of functions \( f_1 \) and \( \varphi \), we compute

\[
f_1(\alpha x + (1 - \alpha) y) - c\varphi(\alpha x + (1 - \alpha) y).
\]

\[
f_1(\alpha x + (1 - \alpha) y) - c\varphi(\alpha x + (1 - \alpha) y) \geq \alpha f_1(x) + (1 - \alpha) f_1(y) - \alpha c\varphi(x) - (1 - \alpha) c\varphi(y) = \alpha[f_1(x) - c\varphi(x)] + (1 - \alpha)[f_1(x) - c\varphi(x)] \geq 0
\]

which is equivalent to \( f(\alpha x + (1 - \alpha) y) \geq c \) and means that \( \alpha x + (1 - \alpha) y \in L_c(f), \ c > 0 \).

Now we are ready to formulate global optimality conditions for problem (2.1). On the other hand, problem (2.1) can be treated equivalently, as quasiconvex maximization problem

\[
\max_{x \in D} (-f) \quad (2.2)
\]

If we apply of Theorem 2.1 [1] to problem (2.2), then we obtain global optimality conditions for the problem in the following proposition.

**Theorem 2.2** Let \( z \) be a global solution to problem (2.1) and \( E_c(f) = \{ y \in \mathbb{R}^n | f(y) = c \} \). Then

\[
\langle f'(y), x - y \rangle \geq 0 \quad \text{for all} \quad y \in E_{f(z)}(f) \quad \text{and} \quad x \in D. \quad (2.3)
\]

If, in addition, \( f'(y) \neq 0 \) holds for all \( y \in E_{f(z)}(f) \), then condition (2.3) is sufficient for \( x \in D \) to be a global solution to problem (2.1)

**Proof.** Direct application of the result of Theorem 2.1 (pp.19, [1]) follows the proof. \( \blacksquare \)

It can be computed that

\[
f'(x) = f' \cdot f_2 + f \cdot f'_2,
\]

where

\[
f'(x) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right).
\]
In order to conclude that a point $z \in \mathbb{D}$ is not a solution to problem (2.1), Theorem (2.1) tells that we need to find $x, y \in \mathbb{R}^n$ such that
\[
\langle f'(y), x - y \rangle < 0, \quad f(y) = f(z), \quad x \in \mathbb{D}.
\]
The following example illustrates use of this property.

**Example 1**
\[
\min_{x \in \mathbb{D}} f = (x_1^2 + 4x_1x_2 + 5x_2^2)(2x_1^2 + 2x_1x_2 + 3x_2^2),
\]
\[
D = \{ x \in \mathbb{R}^2 | -2 \leq x_1 \leq 3, \quad -1 \leq x_2 \leq 4 \}.
\]
We can easily evaluate the gradient of $f$ as:
\[
f' = [(2x_1 + 4x_2)(2x_1^2 + 2x_1x_2 + 3x_2^2) + (x_1^2 + 4x_1x_2 + 5x_2^2)(4x_1 + 2x_2),
\]
\[
(4x_1 + 10x_2)(2x_1^2 + 2x_1x_2 + 3x_2^2) + (x_1^2 + 4x_1x_2 + 5x_2^2)(2x_1 + 6x_2)].
\]
By Lemma 2.1, $f$ is quasiconcave. Now we want to check whether a point $z = (3, -1)$, which is a local minimizer, is a global solution or not. Then consider a pair $u = (-1, 1) \in \mathbb{D}$ and $y = (0, \sqrt{2})$ satisfying $f(y) = f(z) = 30$.

Now we have
\[
f'(y) = (22\sqrt{8}, 60\sqrt{8}) \quad \text{and} \quad \langle f'(y), u - y \rangle = -22\sqrt{8} + (60\sqrt{8})(1 - \sqrt{2}) < 0.
\]
which follows that $z$ is not a global solution. In fact, we can see that the global solution is $z^* = (3, 4)$.

**Conclusion**

We considered the problem of minimizing a product of two concave functions. The problem is nonconvex. We reduced the problem to quasiconvex maximization problem and then applied a result in [1] obtaining new global optimality conditions. Numerical algorithms and results will be discussed in a next paper.
References


