Contractions of stars in bipartite graphs

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Abstract

In this note I prove theorems on reductions of connected bipartite graphs and connected bipartite planar graphs. In both cases, 2-connected families will also be considered. When a set of reductions for a family is concerned it is also shown that each reduction in the set is necessary.

1 Contractions and Minors

One of the first theorems that an undergraduate reader learns in a graph theory course is the characterization of bipartite graphs as graphs containing no odd cycles [4, Theorem 4.7, page 106] or [7, Proposition 1.6.1, page 18]. This is one of the examples of a *good characterization*, in that if a graph is bipartite a bipartition of its set of vertices may be produced efficiently and if it is not bipartite then an odd cycle is found quickly. A computational characterization of bipartite graphs is, however, still desirable: a characterization that directly provides a computational procedure by which all bipartite graphs up to any order may be *computed* and *constructed*. This paper will aim at such a characterization of several classes of

*Work supported by National Science Foundation Fund No.: 10971027 and NRF South Africa.

bipartite graphs. A computational generation of graphs (or other mathematical objects) may be effected by reversal of the operations of reductions. In addition, I also emphisize that the reduction (generation) operations should provide specific minor inclusions. These two criteria are set down by a theorem of Tutte from 1960's on the family of 3-connected graphs [7, Theorem 3.2.5, page 64] and a theorem of Steinitz on triangulations of the sphere [22]. There are a collection of results of this type in the literature.

Connectivity: For the family of 3-connected graphs, the theorem of Tutte is now a theorem in textbooks [7, Theorem 3.2.5, page 64]. The family of 2-connected graphs also has a reduction theorem of this form (see [7, Proposition 3.1.2]). For the family of 4-connected graphs, there are reduction theorems established by many groups of authors (see [9, 10, 15, 16, 17, 20, 21]).

Degree condition: For the family of 3-connected cubic graphs there is the classical theorem of Tutte [23]. The structure of the family of r-connected r-regular graphs for $r \geq 4$ is not known to me. Work in [2] deals with the family of 3-connected graphs with minimum degree not less than 4.

Embedded: For triangulations of the sphere we have the classical theorem of Steinitz [22]. The work on the class of quadrangulations of the sphere include [1, 5, 6, 19], with [5] appearing most recently.

Girth: The paper [13] deals with the family of 3-connected triangle-free (i.e., girth at least 4) graphs.

Let $G = (V, E)$ be a finite undirected simple graph. For $X, Y \subseteq V(G)$, denote

$$[X, Y] = \{xy \in E(G) : x \in X, y \in Y\}.$$

In words, $[X, Y]$ is the set of edges of G with one end in X and the other in Y.

A *contraction* of G is defined to be a partition $\{V_1, V_2, \cdots, V_s\}$ of V such that for each $i = 1, 2, \cdots, s$, the induced subgraph $G|_{V_i}$ is connected. The partition naturally gives rise to a surjective mapping from G to a graph H, also called a *contraction* (graph) of G. The contraction (graph) H is the graph with

$$V(H) = \{V_1, V_2, \cdots, V_s\}, \quad E(H) = \{V_iV_j : i \neq j,\ [V_i, V_j] \neq \emptyset\}.$$

The mapping $f : G \to H$ is called a *contraction* (mapping) from G onto H. If there exists a contraction mapping $f : G \to H$ then G is said to be *contractible* to H. By the definition of H above, H is a simple graph. Extreme but revealing examples of contractions are the one vertex contraction K_1 of any connected graph G and automorphisms of G. In particular, $1 : G \to G$ is a contraction.

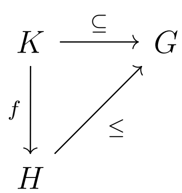
Suppose that $R \subseteq G$ is a connected subgraph. Then the contraction of R in G, denoted G/R, is given by the partition

$$\{V(R), \{v_1\}, \cdots, \{v_m\}\}$$

where $V(G) - V(R) = \{v_i : 1 \leq i \leq m\}$.

Our definition of a contraction agrees with that of Tutte in his theorem ([7, Proposition 3.1.2]). Note that this definition is adequate for directed graphs, infinite graphs and hypergraphs. There are two other variations of a contraction simply by allowing multiple edges (by allowing a number of edges between vertices V_i and V_j that is equal to the size of $[V_i, V_j]$) or loops (by dropping the condition $i \neq j$ in our definition).

A graph H is a *minor* of G, denote $H \leq G$, if G has a subgraph contractible to H. That is, there is a subgraph $K \subseteq G$ and there is a contraction $f : K \to H$. This may be better understood by the following commutative diagram, which comes very handy when proving basic properties of minors.

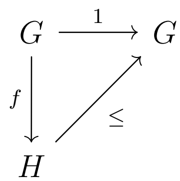


In this diagram each arrow asserts the *existence* of a function. The arrow $K \to G$ is a subgraph inclusion mapping (or embedding of K as a subgraph in G), the arrow $f : K \to H$ is a contraction mapping as defined above, and the arrow $H \to G$ is a *minor inclusion* which completes the diagram.

Examples of basic properties where the diagram above provide easy facility for their proofs are: (1) $\leq$ is a quasi order in a family of graphs; (2) $\leq$ is almost a partial order in families of graphs (meaning that the binary relation $\leq$ is reflixive and transitive). Note that $\leq$ may be regarded as also antisymmetric *if* isomorphic graphs are considered equal (computationally, this depends on the NP-complete problem of isomorphism testing). Under such an assumption, $\leq$ may be considered to be a partial order.

Denote by $\dot{G}$ a subdivision (i.e., a homeomorph) of a graph G. This is usually called a *topological minor*. Some obvious facts concerning minors are: (1) If $H \subseteq G$ then $H \leq G$; (2) If $f : G \to H$ is a contraction, then $H \leq G$; (3) If $\dot{H} \subseteq G$ then $H \leq G$; Note that the converse is not true in general; (4) $(J \leq H) \wedge (H \leq G) \Rightarrow J \leq G$; (5) If $\Delta(G) \leq 3$, then $H \leq G \Leftrightarrow \dot{H} \subseteq G$; (6) If $H \leq G$ and G is planar then H is also planar.

Property (2) above states that if H is a contraction graph of G, then $H \leq G$. The proof of this is direct from the following diagram:



The converse is not true. Let P denote the Petersen graph. Then $K_{3,3} \leq P$ and there exists no contraction $f : P \to K_{3,3}$. Note that there exists a contraction $P \to K_5$.

In this note, properties of graphs and families of graphs are denoted by mathematics script fonts. The reason for this is to facilitate statements of theorems, their proofs and discussions.

Let $\mathscr{P}$ be a property of graphs. A connected subgraph $R \subseteq G$ is said to be *contractible* with respect to $\mathscr{P}$ or *$\mathscr{P}$-contractible* if $G \in \mathscr{P}$ and the contraction $G/R \in \mathscr{P}$. For the class of 3-connected graphs, various results have been obtained in [11] on contractible subgraphs R where $3 \leq |R| \leq 4$. Numerous interesting problems on contractible subgraphs remain open. The reader is referred to [11, 13] and a recent survey [12].

Let $\mathscr{P}$ be a property of graphs. A graph G is said to be *minimal* with respect to $\mathscr{P}$, or *minimally* $\mathscr{P}$, if $G \in \mathscr{P}$ and for each $e \in E(G)$, $G - e \notin \mathscr{P}$. A graph is *$\mathscr{P}$-critical* if $G \in \mathscr{P}$ and for each $x \in V(G)$, $G - x \notin \mathscr{P}$. Thus, for example, a *minimally* k-connected graphs is a graph G that is k-connected but for each $e \in E(G)$, $G - e$ is not k-connected. A graph G is *critically* k-connected if G is k-connected but for each $x \in V(G)$, $G - x$ is not k-connected.

Let G be a connected graph and $S \subseteq V(G)$. If $G - S$ is not connected then S is called a *separator* of G. If $G \neq K_n$ then the least cardinality of a separator is called the *connectivity* (number) of G, and it is denoted by $\kappa(G)$. If $\kappa(G) \geq n$, then G is called an *n-connected* graph. To complete this definition, it is necessary to assert by agreement that $\kappa(K_n) = n - 1$. This is a *negative* approach to life. The *positive* approach is: a graph is k-connected if there exist at least k internally disjoint paths connecting any two vertices of G. The least k such that G is k-connected is defined to be the connectivity of G. Menger's famous theorem [7, Theorem 3.3.1, page 66] asserts that the two definitions are equivalent. For a separator $S \subseteq V(G)$ (not necessarily minimum), the union of at least one but not all components of $G - S$ is called a *fragment* of $G - S$. Thus if F is a fragment of $G - S$, then $G - F - S$ is also a fragment of $G - S$. A minimal fragment is called an *atom* or an *end component*. Note that it is standard to prove that if S is a minimum separator of G, then each vertex of S has a neighbor in every component of $G - S$.

For a graph G and $x \in V(G)$, denote as usual by $N(x) = \{v \in V(G) : vx \in E(G)\}$. Denote $\bar{N}(x) = \{x\} \cup N(x)$.

In this paper, we obtain theorems on reductions of several families of bipartite graphs. Every reduction provides a proper minor inclusion.

2 Connected Bipartite Graphs

In this section, we consider several families of connected simple bipartite graphs and prove theorems on reductions of graphs in these families and on existence of large proper minors in the same family.

First observe that a star contraction provides simple reductions for several families of connected simple bipartite graphs.

THEOREM 2.1. *Let $G \neq K_2$ be a connected bipartite graph. Let $x \in V(G)$ with $k = d(x) = \delta(G)$ and $N(x) = \{x_1, x_2, \cdots, x_k\}$. Then*

$$H = G/G|_{\bar{N}(x)}$$

is a connected bipartite graph.

Proof. Let the bipartition of G be $\{X, Y\}$ with $x \in X$, thus $N(x) \subseteq Y$. Let $f : G \to H$ be the contraction determined by the operation in the definition of H with $y = f(\bar{N}(x)) \in V(H) - V(G)$. Since G is connected and H is contraction of G, H is connected. Let

$$X' = X - \{x\},\ Y' = (Y - \bar{N}(x)) \cup \{y\}.$$

Then $\{X', Y'\}$ is a bipartition of H. Hence H is a connected bipartite graph. □

Note that since G is bipartite $G|_{\bar{N}(x)} \simeq K_{1,r}$ ($r = d(x)$) is a star centered at x with each vertex in $N(x)$ having degree 1. Hence the verbal expression "star contraction".

In Figure 1, we illustrate the reductions used in Theorem 2.1 for the complete bipartite graph $K_{3,3}$, giving a sequence

$$K_{3,3} \longrightarrow K_{1,2} \longrightarrow K_{1,1} \simeq K_2.$$

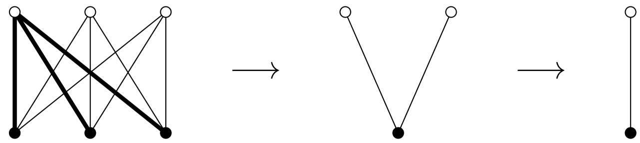


Figure 1: Reductions of bipartite graphs.

THEOREM 2.2. *Let $G \neq K_2$ be a connected bipartite planar graph. Let $x \in V(G)$ with $k = d(x) = \delta(G)$ and $N(x) = \{x_1, x_2, \cdots, x_k\}$. Then*

$$H = G/G|_{\bar{N}(x)}$$

is a connected bipartite planar graph.

Proof. The graph H connected and bipartite as in the proof of Theorem 2.1. In a plane embedding of G, $\bar{N}(x)$ in contained in the interior of a plane region containing no other vertex or edge of G. The new vertex $y \in V(H)$ is embedded in this region. Hence H is planar. Note also that $k \leq 5$. □

For the family of connected bipartite planar graphs, the existence of a large proper minor in the same family is guaranteed.

COROLLARY 2.1. *If G is a connected bipartite planar graph, then there exists a connected bipartite planar graph H such that $H \leq G$ and $|G| - 4 \leq |H| \leq |G| - 1$.*

Proof. Since G is planar and $k = d(x) = \delta(G)$, $k \leq 5$ and hence the assertion. □

The operation in Theorem 2.2 are illustrated using the 3-cube Q_3 and the Herschel's graph in Figure 2. In this figure, the subgraphs $G|_{\bar{N}(x)}$ used in the contraction are drawn in heavy lines. Clearly, contraction of either one of the $K_{1,3}$ in $K_{2,3}$ centered at a black vertex gives K_2.

I have recently obtained a theorem on reductions of 3-connected graphs of minimum degree at least 4. This is a continuation of the work in [2].

Note that there should be a theorem on reductions of connected graphs of minimum degree at least 2 (—this should be fairly straightforward), and a theorem on reductions of 2-connected graphs of minimum degree at least 3 (I have a sketch proof of such a theorem). These, together with my recent results on 3-connected graphs of minimum degree at least 4 makes the initial steps of this study fairly complete.

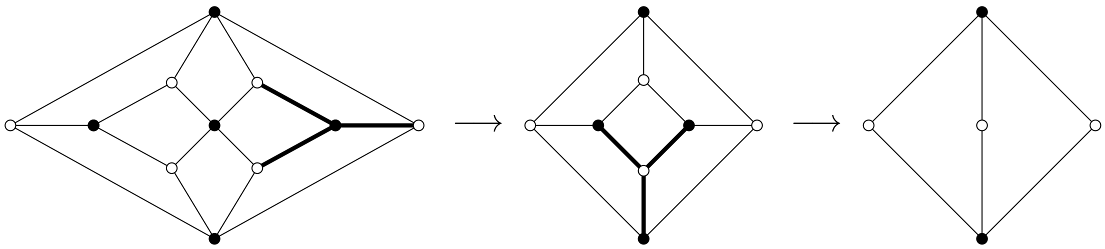


Figure 2: Reductions of bipartite planar graphs.

From now till the end of the section and till the end of the paper, for $r \geq 1$, denote

$$V_r(G) = \{u \in V(G) : d(u) = r\}.$$

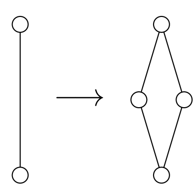
Example 1. For $r \geq 2$, the complete bipartite graph $K_{2,r}$ is minimally 2-connected. For each $e \in E(K_{2,r})$, $K_{2,r}/e$ is 2-connected (i.e., every edge is 2-contractible) but not minimally 2-connected. For each $r \geq 3$ the cycle C_r is minimally 2-connected.

Example 2. Let H be a 2-connected graph and let G be a graph obtained by subdivision of every edge of H with both ends of degree ≥ 3. Then G is minimally 2-connected.

We have

Theorem 2.3. *For each graph H with $|H| = n$ there exists a minimally 2-connected graph G with $|G| \leq n^2$ such that $H \leq G$.*

Proof. Consider K_n. Replace each edge of K_n by a 4-cycle and denote the resulting graph by G:



Then

$$H \subseteq \dot{K}_n \subseteq G.$$

The graph G is 2-connected since every pair of vertices of G is contained in a cycle. It is minimally 2-connected since each edge has an end vertex of degree 2. Clearly,

$$|G| = |K_n| + 2\|K_n\| = n + n(n-1) = n^2.$$

□

Note that not only $H \leq G$ but H is a *topological minor* of G. It is easy to see that n^2 is the minimum order of a minimally 2-connected *universal topological major* for the family of all simple graphs of order $\leq n$.

EXAMPLE 3. Let H be a minimally 2-connected graph, $x, y \in V(G)$ with $d(x), d(y) \geq 3$ and $N(x) \cup N(y) \subseteq V_2(H)$. Let P be a path with

$$V_1(P) = \{x, y\},\ V_2(P) \cap V(H) = \emptyset.$$

Then the graph $G = H \cup P$ is a minimally 2-connected graph.

In all three examples, if G is a minimally 2-connected graph then each edge has an end that is of degree 2. Is this true in general? Dirac's Theorem from 1967 provides an answer to this question.

THEOREM 2.4 (Dirac, 1967). *A 2-connected graph is minimally 2-connected if and only if each edge has an end of degree 2.*

Proof. Sufficiency is clear since if G is a 2-connected graph with each edge having an end of degree 2 then for each $e \in E(G)$, $G - e$ has a vertex of degree 1 and is not 2-connected, and hence G is minimally 2-connected.

Suppose that G is minimally 2-connected. Let $e = xy \in E(G)$ with $d(x), d(y) \geq 3$. Since G is minimally 2-connected, $G - e$ is connected and not 2-connected. Let $\{z\}$ be a separator of $G - e$ such that x and y belong to two components L and R respectively. Consider an edge $ux \in E(G)$ with $u \notin \{y, z\}$. Since $G - ux$ is connected and not 2-connected, there exists a separator $\{w\}$ of $G - ux$ with components A and B containing u and x respectively. Since $d(x) \geq 3$ there is $v \notin \{u, y\}$ such that $vx \in E(G)$. Then $u, v, x \in L$, $y \in R$, $u \in A$ and $v, x \in B$. Hence $v, x \in B \cap L$. But then, in G, $B \cap L \neq \emptyset$ is separated from $\{y\}$ by $\{x\}$. This contradicts the assumption that G is 2-connected. □

This also provides a more constructive result on the family of minimally 2-connected graphs.

THEOREM 2.5. *Let $\mathscr{M}_2$ denote the family of minimally 2-connected graphs. If $G \in \mathscr{M}_2$ and $G \neq K_3$, then either there exists $e = xy \in E(G)$ with $d(x) = d(y) = 2$ such that G/e is minimally 2-connected or there exists $x \in V(G)$ with $d(x) = 2$, $N(x) = \{y, z\}$ and $N(y) \cup N(z) \subseteq V_2(G)$ such that $G - x$ is minimally 2-connected.*

Proof. Let $G \in \mathscr{M}_2 \setminus \{K_3\}$. If there exists $x, y \in V(G)$ with $d(x) = d(y) = 2$ then G/e is a 2-connected graph such that each edge has an end of degree 2. Hence by Theorem 2.4, G/e is minimally 2-connected.

If there exists no pair of adjacent vertices of degree 2 in G, let $x \in V(G)$ with $d(x) = 2$ with $N(x) = \{y, z\}$. Then $d(y), d(z) \geq 3$ and since each edge has an end of degree 2, we have

$$N(y) \cup N(z) \subseteq V_2(G).$$

Now $G - x$ is 2-connected graph with each edge having an end of degree 2. By Theorem 2.4, $G - x$ is minimally 2-connected. □

In [14], Mader proved among other results the following theorem.

Theorem 2.6 (Mader, 1972). *Every minimally k-connected graph has at least $k+1$ vertices of degree k.*

Corollary 2.2. *Every minimally 2-connected graph has at least three vertices of degree 2.*

Lemma 2.1. *If $G \neq K_{2,2}$ is a minimally 2-connected bipartite graph then either there exists $e = xy \in E(G)$ with $d(x) = d(y) = 2$ and $N(y) = \{x, z\}$ such that G/xyz is a minimally 2-connected bipartite graph or there exists $x \in V(G)$ with $d(x) = 2$, $N(x) = \{y, z\}$, $d(y), d(z) \geq 3$ and $N(y) \cup N(z) \subseteq V_2(G)$ such that $G - x$ is a minimally 2-connected bipartite graph.*

Proof. By Theorem 2.5, it suffices to observe that the graphs G/xyz and $G - x$ are both bipartite. □

We now have

Theorem 2.7. *Let $\mathscr{B}_2$ be the family of 2-connected bipartite graphs. If $G \in \mathscr{B}_2$ and $G \neq K_{2,2}$, then one of the following is true:*

(1) *there exists $e \in E(G)$ such that $G - e \in \mathscr{B}_2$;*

(2) *there exists $x \in V(G)$ with $d(x) = 2$, $N(x) = \{y, z\}$, $d(y), d(z) \geq 3$ and $N(y) \cup N(z) \subseteq V_2(G)$ such that $G - x \in \mathscr{B}_2$;*

(3) *$e = xy \in E(G)$ with $d(x) = d(y) = 2$ and $N(y) = \{x, z\}$ such that G/xyz $G/x_1xx_2 \in \mathscr{B}_2$.*

Note that each reduction in the theorem is necessary, that is, independent of the two others. There exist inifinitely many 2-connected bipartite graphs which are not minmally 2-connected, for example those 2-connected bipartite graphs with minimum degree at least 3. For each of these graphs neither the reduction in (2) nor that in (3) may be performed, and hence the reduction in (1) is necessary. Let $r \geq 4$ be even and C_r be the cycle of order r. Then C_r is a minimally 2-connected bipartite graph. Hence the reduction of (1) and the reduction of (3) may not be performed for C_r. The reduction in (3) may not be performed for C_r because no vertex of degree 2 in C_r has a neigbour of degree at least 3. For each graph constructed in Example 2 above, neither a reduction in (1) nor a reduction in (2) may be performed, and hence the reduction in (3) is necessary which may be performed for each of these graphs.

Each of the three reductions in Theorem 2.7 provides a proper minor inclusion. Hence we have

Corollary 2.3. *Let G be a 2-connected bipartite graph. If $G \neq K_{2,2}$ then there exists a 2-connected bipartite graph H such that $H \leq G$ and $|G|-1 \leq |H| \leq |G|-2$.*

We now turn to 2-connected bipartite planar graphs. The following lemma was established in [19] as a simple corollary to Euler's formula for polyhedra.

Lemma 2.2. *Let G be a 2-connected simple bipartite planar graph. Then $\delta(G) = 2$ or 3. If G has no vertex of degree 2, then G has at least 8 vertices of degree 3.*

For the family of 2-connected bipartite planar graphs, we have

Theorem 2.8. *Let $\mathscr{BP}_2$ be the family of 2-connected bipartite planar graphs. If $G \in \mathscr{BP}_2$ and $G \notin \{K_{2,2}, K_{2,3}\}$ then there exists $x \in V(G_i)$ with $N(x) = \{x_1, x_2\}$ such that $H = G/x_1xx_2 \in \mathscr{BP}_2$ or there exists $x \in V(G_i)$ with $N(x) = \{x_1, x_2, x_3\}$ such that $H = (G - xx_1)/x_2xx_3 \in \mathscr{BP}_2$.*

Denote by ρ_1 and ρ_2 respectively the reductions in (1) and (2) in the conclusion of the theorem.

Proof. Let G be a 2-connected bipartite planar graph. Then by Lemma 2.2, $\delta(G) = 2$ or 3. Suppose that $G \notin \{K_{2,2}, K_{2,3}\}$.

If $\delta(G) = 2$, then let $x \in V(G)$ with $d(x) = 2$ and let $N(x) = \{x_1, x_2\}$. Then $H = G/x_1xx_2$ is a connected bipartite planar graph. (Note that this is $\rho_1 : G \mapsto H$.) It will be shown that H is 2-connected. Let $f : G \to H$ be the contraction determined by the operation, with $f(\{x, x_1, x_2\}) = x'$. Then

$$\begin{aligned} V(H) &= (V(G) - \{x, x_1, x_2\}) \cup \{x'\}, \\ E(H) &= (E(G) - \{xx_1, xx_2\}) \cup [x', N(x_1) \cup N(x_2) - \{x, x_1, x_2\}]\,. \end{aligned}$$

Let $u, v \in V(H)$. If $x' \notin \{u, v\}$ then let P and Q be two internally disjoint (u, v)-paths in G. If $x \notin V(P) \cup V(Q)$ then P and Q are two internally disjoint paths of H connecting u and v. If $x \in V(P) \cup V(Q)$ then since P and Q are internally disjoint, it may be assumed that $x \in V(P)$ and $x \notin V(Q)$. Suppose that $z_1x_1xx_2z_2 \subset P$. Then $P' = (P - \{x, x_1, x_2\}) \cup \{z_1x'z_2\}$ and Q are two internally disjoint (u, v)-paths in H. If $x' \in \{u, v\}$ then let $u = x'$. Sincce G is 2-connected there are two disjoint (x, v)-paths P and Q in G. Now x is a common end of the two paths P and Q. Assume that $z_1x_1x \subseteq P$ and $z_2x_2x \subseteq Q$. Then

$$P' = (P - \{x, x_1\}) \cup \{x'z_1\},\ Q' = (Q - \{x, x_2\}) \cup \{x'z_2\}$$

are two internally disjoint (x', v)-paths in H. Hence we have shown that H is 2-connected.

Assume, therefore, that $\delta(G) = 3$ and let $x \in V(G)$ with $N(x) = \{x_1, x_2, x_3\}$. Let $H = (G - xx_1)/x_2xx_3$. (Note that this is $\rho_2 : G \mapsto H$.) Then H is a connected planar graph. It will be shown that H is bipartite and 2-connected.

Let $f : G - xx_1 \to H$ be the contraction determined by operation ρ_2, with $x' = f(\{x, x_2, x_3\})$. Let X and Y be the parts in the bipartition of G with $x_2, x_3 \in X$ and hence $x \in Y$. Then $(X - \{x_2, x_3\}) \cup \{x'\}$ and $Y - x$ are parts in a bipartition of H. This is because

$$\begin{aligned} V(H) &= (V(G) - \{x, x_2, x_3\}) \cup \{x'\}, \\ E(H) &= (E(G) - \{xx_j : j = 1, 2, 3\}) \cup [x', N(x_2) \cup N(x_3) - \{x\}]. \end{aligned}$$

Hence H is bipartite.

Assume that H is not 2-connected. Since H is connected, let $\{z\}$ be a separator of H. If $z \neq x'$ then $\{z\}$ would be a separator of G, a contradiction.

Assume therefore that $z = x'$. Let L be a component of $H - z$ containing x_1 and let $R = H - x' - L$. Now since $d(x_1) \geq 3$, $d_H(x_1) \geq 2$ and $\{x_1\}$ is a separator of G. This is a contradiction. Hence we have shown that H is 2-connected.

The proof is complete. □

That the first reduction is necessary is seen from circuits of even length. These are 2-connected simple bipartite planar graphs regular of degree 2. That the second reduction is necessary may be seen from the existence of connected (and hence 2-connected) cubic bipartite planar graphs.

Corollary 2.4. *If G is a simple connected bipartite planar graph and $|G| > 5$ then there exists a simple connected bipartite planar graph H such that $H \leq G$ and $|H| = |G| - 2$.*

Note that $K_{2,2} \subset K_{2,3}$ and hence $K_{2,2} < K_{2,3}$. Thus if G is a 2-connected bipartite planar graph then $K_{2,2} \leq G$.

A *homomorphism* $f : G \to H$ is a mapping $f : V(G) \to V(H)$ such that $xy \in E(G) \Rightarrow f(x)f(y) \in E(H)$. An *r-fold covering* is a homomorphism $f : G \to H$ such that for each $e \in E(H)$, $|f^{-1}(e)| = r$. Note that $f^{-1}(e) \subseteq E(G)$.

If G is a 2-connected bipartite planar graph then is there a homomorphism $f : G \to K_{2,2}$? Note that G is a bipartite graph iff there exists a homomorphism $f : G \to K_2$. Note also that there is a homomorphism $f : Q_3 \to K_{2,2}$ as a 3-fold covering, and there is a homomorphism $g : H \to K_{2,2}$ where H is the Herschel's graph.

Acknowledgement. I would like to record my thanks to Professor R. Enkhbat, director of the Mongolian Mathematics Society for his invitation to the Fourth International Conference on Optimization, Simulation and Optimal Control, Ulaabaatar July 1-4, 2013.

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