Z-folding and its applications

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Abstract

As is known, proper transformation is very attractive and useful in solving some delicate problems. In this note we propose a folding algorithm for solving the pentadiagonal system of linear equations, which allows us to reduce the system by solving two tridiagonal systems sequentially. As applications, the proposed folding algorithm is used to solve some concrete systems of linear equations.
Key words: linear transform, linear equations, boundary conditions.

1 Introduction

The system of linear equations

$$u_{i-2} + du_{i-1} + cu_i + du_{i+1} + u_{i+2} = f_i, \quad i = 2(1)N-2, \tag{1}$$

with corresponding boundary conditions is often appeared in computational practice and solved by LU decomposition method [3]. In this note we propose a simple algorithm for solving (1). We use the linear transform

$$z_i = u_{i-1} + au_i + u_{i+1}, \quad i = 1, \dots, N-1 \tag{2}$$

to reduce (1) and consider next relation

$$z_{i-1} + bz_i + z_{i+1} = u_{i-2} + (a+b)u_{i-1} + (2+ab)u_i + (a+b)u_{i+1} + u_{i+2}. \tag{3}$$

If we choose a and b in (2), (3) such that

$$\begin{aligned} a+b&=d,\\ 2+ab&=c \end{aligned} \tag{4}$$

then from (3) it clear that

$$z_{i-1}+bz_i+z_{i+1}=f_i, \quad i=2(1)N-2. \tag{5}$$

Thus, we have

Theorem 1. *Let u_i be the solution to (1) and*

$$d^2-4(c-2)\geq 0. \tag{6}$$

Then z_i given by (2) will be the solution of system (5) under conditions (4) or

$$b_{1,2}=\frac{d\pm\sqrt{d^2-4(c-2)}}{2}, \quad a_{1,2}=\frac{d\mp\sqrt{d^2-4(c-2)}}{2}. \tag{4'}$$

Proof. The proof is obvious from (2), (3), (4) and (5). □

By virtue of theorem 1, the system of equations (1) leads to two simple tridiagonal systems (5) and (2), which are solved by the efficient elimination method. In connection with this we will call the transformation (2) the **Z**-*folding*. In order to solve the tridiagonal systems (5) and (2) boundary conditions are needed, which either may be followed from the boundary conditions for system (1), or by another ways.

2 Applications

We consider next system

$$m_{i-2}+56m_{i-1}+246m_i+56m_{i+1}+m_{i+2}=b_i, \quad i=2(1)N-2, \tag{7}$$

where $b_i=\frac{30}{h^2}(-I_{i-1}-9I_i+9I_{i+1}+I_{i+2})$. Such system arose in constructing integro quintic splines [2]. In (7) $m_i=\alpha_i, \quad i=0,1,N-1,N$ to be known.

Therefore (7) is $N-3$ by $N-3$ linear penta-diagonal equations to obtain a unique solution set for $N-3$ remaining parameters $m_2,m_3,\ldots,m_{N-2}$.

The system (7) is a particular case of (1) with

$$d=56, \quad c=246.$$

Therefore it can be reduced to two system of (5) and (2) with coefficients

$$b_{1,2} = 28 \mp 6\sqrt{15}, \quad a_{1,2} = 28 \pm 6\sqrt{15}. \tag{8}$$

Using Taylor expansions for quintic spline $Q(x)$ and its first derivative $m_i = Q'(x_i)$ we obtain [6]

$$\begin{aligned} z_1 &= m_0 + am_1 + m_2 = (2+a)m_1 + h^2 n_1 + O(h^4), \quad n_i = Q'''(x_i), \\ z_{N-1} &= m_{N-2} + am_{N-1} + m_N = (2+a)m_{N-1} + h^2 n_{N-1} + O(h^4), \end{aligned} \tag{9}$$

which is used as boundary conditions for (5), when m_1, n_1 and m_{N-1}, n_{N-1} are given.

According to (8) the system (5), (9) has a diagonal dominance and consequently it is solved by efficient elimination method. When $z_1, z_2, \ldots, z_{N-1}$ are known, then the system (2) is also solved by efficient elimination method by virtue of (8) for $i = 2, \ldots, N-2$ [6].

As a second example, we consider the system

$$Ac = F, \tag{10}$$

where

$$A = \begin{pmatrix} 1 & 11 & 11 & 1 & 0 & & & & \\ 0 & 1 & 11 & 11 & 1 & 0 & & & \\ 1 & 26 & 66 & 26 & 1 & 0 & & & \\ & 1 & 26 & 66 & 26 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 26 & 66 & 26 & 1 \\ & & & 1 & 11 & 11 & 1 & 0 \\ & & & 0 & 1 & 11 & 11 & 1 \end{pmatrix}_{(n+4)\times(n+4)}$$

and $c = (c_{-2}, c_{-1}, c_0, c_1, \ldots, c_n, c_{n+1})^T$,
$F = (24y_0, 24y_1, \frac{120}{h} I_0, \ldots, \frac{120}{h} I_{n-1}, 24y_{n-1}, 24y_n)^T$.

Such system arose in constructing quartic spline $S(x)$ such that [4]

$$\int_{x_i}^{x_{i+1}} S(x)dx = I_i = \int_{x_i}^{x_{i+1}} y(x)dx, \quad i = 0, 1, \ldots, n-1.$$

The system (10) is also a particular case of (1) with $d = 26$, $c = 66$.

Therefore, it can be reduced to two diagonally dominant system of (5) and (2) with coefficients

$$a_{1,2} = 13 \mp \sqrt{105}, \quad b_{1,2} = 13 \pm \sqrt{105}. \tag{11}$$

The system of equations (10) in term of z_i leads to

$$\begin{array}{l} z_0 + z_1 = (2a+4)y_1 - \frac{a-10}{6}h^2 y_1'', \\ z_{i-1} + bz_i + z_{i+1} = \frac{120}{h}I_i, \quad i = 0, 1, \ldots, n-1, \\ z_{n-2} + z_{n-1} = (2a+4)y_{n-1} - \frac{a-10}{6}h^2 y_{n-1}'', \end{array} \tag{12}$$

where $y_i, \; y_i''$ are to be known for $i = 1, n-1$.

The matrix of system (12) is strictly diagonally dominant. Hence the system (12) has a unique solution set $(z_{-1}, z_0, \ldots, z_{n-1})$. The coefficients of quartic spline $S(x)$ is determined by solving system

$$c_{i-1} + ac_i + c_{i+1} = z_i, \quad i = -1, 0, \ldots, n.$$

More precisely, it is known that

$$\begin{array}{l} c_0 + c_1 = 2s_1 - \frac{h^2}{6}s_1'', \\ c_{n-1} + c_n = 2s_{n-1} - \frac{h^2}{6}s_{n-1}''. \end{array}$$

Then we obtain a system

$$\left.\begin{array}{l} (a-1)c_1 + c_2 = z_1 - 2y_1 + \frac{h^2}{6}y_1'', \\ c_{i-1} + ac_i + c_{i+1} = z_i, \quad i = 2, \ldots, n-2 \\ c_{n-2} + (a-1)c_{n-1} = z_{n-1} - 2y_{n-1} + \frac{h^2}{6}y_{n-1}''. \end{array}\right\} \tag{13}$$

After solving the last system we find

$$\begin{array}{ll} c_0 = 2y_1 - \frac{h^2}{6}y_1'' - c_1, & c_n = 2y_{n-1} - \frac{h^2}{6}y_{n-1}'' - c_{n-1}, \\ c_{-1} = z_0 - ac_0 - c_1, & c_{-2} = z_{-1} - ac_{-1} - c_0, \\ c_n = 2z_{n-1} - c_{n-2} - ac_{n-1}, & c_{n+1} = z_n - c_{n-1} - ac_n. \end{array} \tag{14}$$

Thus, all coefficients of (10) are determined. It should be mentioned that the system similar to (10) appeared also for constructing quintic spline interpolation [1] and for numerical solution of Burgers' equation [5].

As a third examples, we consider the two-point boundary value problem

$$u'' + p(x)u' + q(x)u = f(x), \quad x \in [a, b] \tag{15}$$

subject to boundary conditions

$$u(a) = \gamma_1, \quad u(b) = \gamma_2, \tag{16}$$

where γ_1 and γ_2 are given constants. Well known that the boundary value problem (15), (16) has a unique solution if $q(x) \le q < 0$ and $u \in C^{k+2}[a,b]$ under conditions $p, q, f \in C^k[a,b]$. For numerical solution of problem (15), (16) we use the following five-point centered difference approximations of order $O(h^4)$

$$\begin{aligned} u_i'' &\approx \tfrac{1}{6h^2}(-u_{i-2} + 10u_{i-1} - 18u_i + 10u_{i+1} - u_{i+2}), \\ u_i' &\approx \tfrac{1}{12h}(u_{i-2} - 8u_{i-1} + 8u_{i+1} - u_{i+2}), \\ u_i &\approx \tfrac{1}{36}(-u_{i-2} + 4u_{i-1} + 30u_i + 4u_{i+1} - u_{i+2}). \end{aligned} \tag{17}$$

Assume that the solution of the problem (15), (16) is extended outside the interval [a,b]. More precisely, we assume that $u \in C^{k+2}[a-2h, b+2h]$.

Then substituting (17) into (15) and setting $x = x_i$, we obtain

$$\begin{aligned} &(1 - \tfrac{h}{2}p_i + \tfrac{h^2}{6}q_i)y_{i-2} + (-10 + 4hp_i - \tfrac{2}{3}h^2q_i)y_{i-1} + (18 - 5h^2q_i)y_i + \\ &+(-10 - 4hp_i - \tfrac{2}{3}h^2q_i)y_{i+1} + (1 + \tfrac{h}{2}p_i + \tfrac{h^2}{6})y_{i+2} = -6h^2 f_i, \\ &i = 0(1)N. \end{aligned} \tag{18}$$

It is easy to show that the last system of equation (18) has not diagonal dominance as well as constant coefficients. Nevertheless the **Z**-*folding algorithm* works well for this system. Indeed by the following transformation

$$z_i = y_{i-1} - 8y_i + y_{i+1}, \quad i = -1, 0, \ldots, N+1 \tag{19}$$

the equation (18) leads to

$$A_i z_{i-1} - C_i z_i + B_i z_{i+1} = -6h^2 f_i, \quad i = 0(1)N, \tag{20}$$

where

$$\begin{aligned} A_i &= 1 - \tfrac{h}{2}p_i + \tfrac{h^2}{6}q_i, \\ B_i &= 1 + \tfrac{h}{2}p_i + \tfrac{h^2}{6}q_i, \\ C_i &= 2 - \tfrac{2}{3}h^2 q_i. \end{aligned} \tag{21}$$

In order to determine the unknowns $z_{-1}, z_0, \ldots, z_N, z_{N+1}$ in (20) we need two additional equations. To obtain these equations we use (19) and the following relation

$$z_{i-1} + 4z_i + z_{i+1} = -36y_i + h^4 y_i^{(4)} + O(h^6), \quad i = 0, N \tag{22}$$

which obtained from Taylor expansion for $y(x)$.

From (22) it is clear that, without loss of accuracy, one can consider the following equations

$$z_{i-1} + 4z_i + z_{i+1} = -36y_i, \quad i = 0 \ \texttt{ and } \ i = N \tag{23}$$

as additional two equations, when y_0 and y_N are given.

We can eliminate z_{-1} and z_{N+1} from (20) and (23) for $i = 0$ and $i = N$ respectively. As a result we have

$$\begin{array}{l} (4A_0 + C_0)z_0 + (A_0 - B_0)z_1 = -36A_0\gamma_1 + 6h^2 f_0, \\ A_i z_{i-1} - C_i z_i + B_i z_{i+1} = -6h^2 f_i, \quad i = 1, 2, \ldots, N-1, \\ (B_N - A_N)z_{N-1} + (4B_N + C_N)z_N = -36B_N\gamma_2 + 6h^2 f_N. \end{array} \tag{24}$$

It is easy to show that the matrix of the system (24) has diagonal dominance for small enough h. Therefore it has a unique solution set $(z_0, z_1, \ldots, z_N)$ and it can be solved by efficient elimination method. After solving (24) we can solve the system

$$\begin{array}{rl} y_0 = \gamma_1, & \\ y_{i-1} - 8y_i + y_{i+1} = z_i, & i = 1, 2, \ldots, N-1, \\ y_N = \gamma_2 & \end{array} \tag{25}$$

and find $y_i, \quad i = 1, 2, \ldots, N-1$.

3 Conclusions

In this note we propose **Z**-*folding algorithm* for solving pentadiagonal system of linear equations, which allows us to reduce the system to solve two tridiagonal systems sequentially. It is easy to show that this transformation is equivalent to represent the matrix of system (1) as a factorization of two tridiagonal matrices, i.e.,

$$A = BC,$$

which allows us to save both the arithmetic operations and CPU time. This algorithm is especially advantageous for the pentadiagonal system of equations to be solved at each time step [5].

References

[1] L.Bataa. *The quintic spline interpolation*, Master thesis, National University of Mongolia, Ulaanbaatar, 2000.

[2] H.Behforooz, *Interpolation by integro quintic splines*, Appl. Math. Comput. 216(2010) 364-367.

[3] J.W.Demmel, *Applied Numerical linear algebra, SIAM*, Philadelphia, 1997.

[4] F.G.Lang, X.P.Xu, *On integro quartic spline interpolation*, Comput. Appl. Math. 236(2012) 4214-4226.

[5] T.Zhanlav, V.Ulziibayar, *High-order numerical solution of Burgers' equation*, International Journal of Mathematical Sciences. 33(2013), 1374-1378.

[6] T.Zhanlav, R.Mijiddorj, *The integro quintic spline and its approximation properties.* Submitted Appl. Math. Comput.