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Abstract: Mining is viewed as an important engine for Mongolia’s continued economic growth. One of the key concerns for any developing and transitional state is how quickly and effectively industrial and infrastructure investments translate into widespread economic development. This paper evaluates Mongolian policies towards the mining sector and their effectiveness as a catalyst for sustained economic growth. Of particular interest is the question of how a policy centered on extractive industries such as coal and other energy materials as well as copper and potential rare earths fit into the growing global economy.

Keywords: Mongolia, mining, economic development, optimal growth

1. Introduction

The Mongolian mining industry is viewed as an important engine for Mongolia’s continued economic growth. Policies enacted over the past few years provide that all Mongolian citizens receive payments from and stock (non-tradable) in mining ventures such as Erdenet MGL and Oyu Tolgoi. One of the key concerns for any developing and transitional state is how quickly and effectively industrial and infrastructure investments translate into widespread economic development. This paper evaluates Mongolian policies towards the mining sector against other states that have relied on natural resource exploitation as a vehicle of economic growth. Of particular interest is the question of how a policy centered on extractive industries and in particular coal and copper mining fit into the growing global economy.

Mining and its associated activities represent approximately 30 percent of Mongolia’s total GDP and close to 80 percent of the country’s exports. With such a high degree of reliance on commodity exports, the country’s overall economic well-being is very sensitive to fluctuations in commodity prices. Batgelder[3] reports that
the Mongolian economy was adversely affected by the global recession. In particular commodity prices and consequently funding for a number of social welfare and infrastructure projects was negatively impacted. The country’s trade deficit soared leading to both currency devaluation and inflationary pressures.

Radetzki and Van Duyne [17] suggest that given the high level of investment required to bring mining facilities into production and the length of time required for firms to recoup their investments, that when faced with slowdowns in growth rates, the industry will be characterized by excess capacity and low prices. Their evaluation is based upon an optimum control model. They conclude that profit maximizing firms will take projects through to their full fruition even when faced with decreasing commodity prices. Dependency upon commodity production is risky as anticipated revenues to fund growth may not be realized and could result in a deterioration of a country’s terms of trade.

Batchulun and Lin [2] report that the mining sector is responsible for 50 percent of total state tax revenues. The mining sector accounts for almost 12 percent of total employment, split evenly with 100,000 people engaged in (informal sector) artisanal mining, and 100,000 workers employed by formal sector mining companies. Of considerable importance to their analysis is the level of infrastructure in place to allow mine operators to become competitive and successful in the marketplace. Significant reserves of coal and copper lie along just north of the China-Mongolian border. The infrastructure in place such as access to water, electric power and transportation services is limited. There are also additional concerns regarding the extent to which mining sector interests have fueled government corruption. They conclude though, that the growth of the mining sector represents the best possible scenario for sustaining Mongolian economic growth as it spurs the development of new infrastructure and as a source of revenue to develop other industrial sectors.

Applying the concept of export-led growth (ELG) to Chile, Silverstovs and Herzer [19] utilize a cointegration, error correction, and Grainger causality model to evaluate ELG. They find that product mix does matter and in the case of Chile, manufacturing exports lead to productivity enhancing effects. Commodity production such as mining on the other hand was more likely to result in productivity limiting effects.

Slack [20] argues that any policy relying on mining as a vehicle for long run sustainable growth needs to be fully evaluated. While there do appear to be some late nineteenth and early twentieth century success stories such as Canada and the United States, an important part of that success is in the institutional and industrial setting that existed at that point in time. There is less evidence of successful and
sustainable growth in developing countries such as Chile, Botswana, and Ghana. While some growth has occurred, income inequality has been exacerbated. Mining tends to place strains on environmental factors such as water resources and land-use policies. Employment multipliers very often fail to materialize as the industry is very capital intensive and relies on imported capital and other inputs. In addition, the downstream in-country intermediate and final processing of ore also tends to be limited given the ease at which unprocessed ore can be shipped to global markets.

Hilson and Maconachie [12] raise arguments related to the “resource curse”, where countries rich in natural resources may fail to achieve sustainable growth. Underlying the problem are issues related to poor governance, corruption, and the assurance that funds generated by the mining sector are utilized to propel economic growth. They cite numerous examples of underperforming African states. Much of their analysis focuses upon the Extractive Industry Transparency Initiative (EITI) which establishes a process for countries with a high reliance on the exploitation of local natural resources to establish a more transparent and open process in the use of the revenue streams from this sector.

Dane [5] on the other hand argues that mining has helped to produce some modest gains in economic growth and raising the standards of living for residents in the communities surrounding Lando Colliery. Thus mining has had a positive impact in the region – though there are still lingering issues to be addressed.

Rolfe et al. [18] similarly argue that Australia, and in particular the Bowen Basin has experienced positive growth associated with the rising coal industry. They too make the case that the Bowen Basin region has had to shoulder an extensive burden of potential negative externalities that can arise from the mining industry especially since a significant portion of the work force was brought in from elsewhere in and out of the country. This in turn has impacted the use of local facilities, led to shortages of housing and other infrastructure, and again limited gains within the community.

While Navch et al. [16] document a number of the issues and problems arising from independent informal gold mining operations, a recent UNIDO [22] report focuses upon establishing a plan for creating sustainable economic growth in Mongolia. The UNIDO document in particular develops a blueprint for the county’s long term economic growth and calls for taking the revenues flowing from mining and placing them into a sovereign wealth fund to be used to finance other industrial development within the country. An underlying concern is the fact that given the population of the country at approximately 2.7 million, and the vast expanse of the country itself, it will be difficult to develop sustainable manufacturing and other
facilities that will be able to achieve and operate at scale economies in the global economy.

The rest of this paper is organized as follows. In the next section of this paper is a discussion of the methodology that will be used to analyze growth attributable to mining sector development. The data and analysis are presented in Section 3. Section 4 presents the conclusions of the analysis.

2. The Model

The effects and influence of the Mongolian mining industry on economic growth are evaluated through the use of a modified production function specified as:

\[ Y_t = AK_t^{\alpha} L_t^{\beta} \]  

(1)

where \( Y \) represents output, \( A \) is total factor productivity, \( K \) is capital, and \( L \) is labor. Following Silverstov and Herzer [19], total factor productivity \( A \), can be further defined in terms of various components, in this case we can specify it as merchandise exports \( MX \), minerals exports \( XM \), and other exogenous inputs \( C \). Substituting this into equation 1 yields:

\[ Y_t = C_t K_t^{\alpha} L_t^{\beta} MX_t^\gamma XM_t^\delta \]  

(2)

In log form, this equation becomes:

\[ \log Y_t = \log C_t + \alpha \log K_t + \beta \log L_t + \gamma \log MX_t + \delta \log XM_t + \epsilon_t \]  

(3)

and can now be estimated where, \( \alpha, \beta, \gamma \) and \( \delta \) are the constant elasticities of output, and \( \epsilon \) is an error term. In order to properly estimate this function and fully evaluate the effects of exports on output or in other words the export led growth hypothesis, both merchandise exports and mineral exports must be netted out of output. Subtracting merchandise exports and mineral experts from \( Y \) yields, \( NY \), and the estimating equation becomes:

\[ \log NY_t = \log C_t + \alpha \log K_t + \beta \log L_t + \gamma \log MX_t + \delta \log XM_t + \epsilon_t. \]  

(4)

Theory would suggest that the coefficients on \( K \) and \( L \) should be positive, or in other words increases in capital and labor would positively affect output. If export industries are having positive spillovers on the rest of the economy, then we would
also expect the coefficients on MX and XM to be positive. These spillovers are usually the result of the need for export industries to adopt production methods that would allow them to compete in international markets. This may result in the introduction of new technologies, and increased productivity that spills over into other industries and parts of a country’s infrastructure.

At issue in this analysis is the question of causality. Statistically, causality can be evaluated using an error correction model which embodies the notion of Granger Causality. One of the problems that regularly arises with time series data is that the variables being used in the analysis change over time, or in other words is non-stationary. In the event that the variables are not stationary, the ordinary least squares regression of equations 4 would give rise to potentially spurious results. The error correction model resolves this issue, thus Equation 4 is estimated in the following form:

$$\Delta \log NY_t = \Delta \log C_{t-1} + \alpha \Delta \log K_{t-1} + \beta \Delta \log L_{t-1} + \gamma \Delta \log MX_{t-1} + \delta \Delta \log XM_{t-1} + \epsilon_t. \quad (5)$$

3. Data and analysis

The data for the analysis comes from the World Development Indicators from the World Bank database for the period 1980 through 2011. Output is measured as real GDP denoted in U.S. constant dollars (indexed to 2000). Capital (K) in the analysis was proxied by gross fixed capital formation (constant dollars). Labor statistics for Mongolia were only available from 1991 forward – and thus did not provide a long enough series to undertake a regression analysis. Total population was available for the entire period under analysis, and thus was used as a proxy for labor force (L). Merchandise exports (MX) were derived by using the percentage of merchandise exports from total output to calculate total merchandise exports in constant U.S. dollars.

Similarly, mineral exports (XM) were proxied for using Mineral Rents as a percent of GDP, and recorded in constant dollars. One additional variable is included in the analysis – a dichotomous dummy variable (Soviet) to account for the collapse of the Soviet Union in 1991, which impacted the structure of Mongolia’s institutional, political and trade relationships and ushered in the transition to a market based economy. This variable takes on the value of ‘0’ prior to 1991, and ‘1’ from 1991 forward.
Mongolia has undergone a vast transformation over the past 30 years from a socialist state trading primarily with the Soviet Union to a market oriented economy. Between 1981 and 2010, the economy grew at an average rate of 3.84 percent annually (Figure 1). The dominant exports for the country tend to be primary products with mineral products ranging from a low of 29 percent in 2001 to a high of 80 percent of total exports in 2010 (Figure 2). This is followed by textiles, and natural or cultured stones, and precious metals. Mining exports represents an important source of foreign exchange and revenue for the country. Making these products available to the international market though requires extensive investment in infrastructure and technology. Thus, the mining industry represents a potentially important component of an export-led growth strategy.

Figure 2: Annual Growth Rate
Evaluating the error correction model requires several steps. In the first step, all of the variables must be evaluated to establish that they are stationary and that there is at least one cointegrating relationship. Applying the augmented Dickey-Fuller test, all of the variables are found to be I(1). Further examination of the data does establish that there is at least one cointegrating relationship among the variables. Summary statistics are presented in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>NY</th>
<th>K (Population)</th>
<th>MX</th>
<th>XM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>6370000000.00</td>
<td>4160000000.00</td>
<td>2285681.00</td>
<td>3120000000.00</td>
</tr>
<tr>
<td>Median</td>
<td>6200000000.00</td>
<td>4190000000.00</td>
<td>2315750.00</td>
<td>2700000000.00</td>
</tr>
<tr>
<td>Maximum</td>
<td>8770000000.00</td>
<td>7350000000.00</td>
<td>2756000.00</td>
<td>5440000000.00</td>
</tr>
<tr>
<td>Minimum</td>
<td>3820000000.00</td>
<td>1660000000.00</td>
<td>1736994.00</td>
<td>1110000000.00</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>1290000000.00</td>
<td>1610000000.00</td>
<td>280980.70</td>
<td>1240000000.00</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.03</td>
<td>0.21</td>
<td>-0.33</td>
<td>0.18</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>2.44</td>
<td>1.94</td>
<td>2.26</td>
<td>1.75</td>
</tr>
</tbody>
</table>

Table 1: Descriptive Statistics (30 observations)

Ideally, the error correction model should be evaluated as a VAR. However, given the limited amount of data available for the analysis, the models proved to be inconsistent. An initial evaluation of the model using both a one step procedure and the Engle-Granger two-step procedure were applied to the data however, and
yielded some preliminary results. Overall, these results appear to be consistent with the signs as predicted in the discussion of the model in Section 2. Only the results of the one step procedure are reported here.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>-28.82473</td>
<td>11.55541</td>
<td>-2.494479</td>
<td>0.0226</td>
</tr>
<tr>
<td>DLOG(K)</td>
<td>0.017394</td>
<td>0.232926</td>
<td>0.074676</td>
<td>0.9413</td>
</tr>
<tr>
<td>DLOG(L)</td>
<td>17.61148</td>
<td>22.61938</td>
<td>0.778602</td>
<td>0.4463</td>
</tr>
<tr>
<td>DLOG(MX)</td>
<td>-0.249887</td>
<td>0.174568</td>
<td>-1.43146</td>
<td>0.1694</td>
</tr>
<tr>
<td>DLOG(XM)</td>
<td>-0.286266</td>
<td>0.108238</td>
<td>-2.644773</td>
<td>0.0165</td>
</tr>
<tr>
<td>SOVIET</td>
<td>-0.269948</td>
<td>0.144891</td>
<td>-1.863112</td>
<td>0.0788</td>
</tr>
<tr>
<td>LOG(K(-1))</td>
<td>0.02677</td>
<td>0.269408</td>
<td>0.099368</td>
<td>0.9219</td>
</tr>
<tr>
<td>LOG(L(-1))</td>
<td>3.677676</td>
<td>1.212513</td>
<td>3.033103</td>
<td>0.0072</td>
</tr>
<tr>
<td>LOG(MX(-1))</td>
<td>-0.249222</td>
<td>0.123412</td>
<td>-2.019424</td>
<td>0.0586</td>
</tr>
<tr>
<td>LOG(XM(-1))</td>
<td>-0.224687</td>
<td>0.093561</td>
<td>-2.401509</td>
<td>0.0273</td>
</tr>
<tr>
<td>LOG(NY(-1))</td>
<td>-0.821425</td>
<td>0.14627</td>
<td>-5.615821</td>
<td>0.0000</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.726788</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adjusted R-squared</td>
<td>0.575004</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S.E. of regression</td>
<td>0.135296</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum squared resid</td>
<td>0.329492</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log likelihood</td>
<td>23.77452</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean dependent var</td>
<td>0.005959</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S.D. dependent var</td>
<td>0.207536</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Akaike criterion</td>
<td>-0.881002</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Schwarz criterion</td>
<td>-0.362372</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F-statistic</td>
<td>4.788302</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Prob(F-statistic)</td>
<td>0.001974</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Estimates – One Step ECM, dependent variable Dlog(NY)

The regression results shown in Table 2 consist of two basic components, the changes component that evaluates short term adjustments represented by the differenced values of NY, K, L, MX, and XM, and the error correction components shown as lagged values of the these variables. A dummy variable Soviet is included in the analysis to account for changes that occurred in Mongolia’s economy and trade regime following the collapse of the Soviet Union in 1991. The coefficient on lagged output is of critical importance as it represents the rate at which the deviations from trend growth are reversed. It is reported here as both negative and significant indicating that there is a cointegrated relationship between the variables and it is stable. While the signs on both K and L are positive, in first difference neither coefficients are significant. The signs on the export values are all found to be negative, and in the case of mineral exports, the coefficients are significant for both
first difference and lagged estimates. This suggests that growth in the mining sector does not lead to significant spillovers in other sectors of the economy. This finding is similar to Silverstov and Herzer’s [19] analysis of the Chilean mining sector where they found that manufacturing exports led to export-led growth, but that the mining sector did not.

4. Conclusions

While the analysis presented above is still preliminary, it does suggest that mining sector has not in and of itself produced export-led growth. The findings of the analysis are also suggestive that as a whole, Mongolia’s export sector will need to undergo significant transformation before export-led growth becomes apparent. One issue to keep in mind at this point is that exports as a whole are predominantly in primary production – mining, mining related industries, animal based products, and textiles. This does not mean that mining is not important to the economy. As a matter of fact, mining and mining exports are still the largest component of exports and of critical importance to the country’s long run growth plan.

Although positive spillovers from foreign investment in the industry have not fully developed, the industry generates a tremendous stream of revenue into the country. A revenue stream that a recent United Nations [22] report suggests should be tapped to create a sovereign wealth fund to fund infrastructure and industrial development. The results of the preliminary analysis presented here provide additional support for this policy.

References

The problem of control constructing, which provides the implementation of the state and mixed constraints under persistent disturbances

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Abstract The article is devoted to the construction of algorithmic support of problem solution. This problem is an optimal control problem for linear dynamic system under given deterministic perturbations. Here the main purpose of control is to keep the system in the state and mixed control-state constraints with a minimum expenditure of energy. We consider an auxiliary optimal control problem with quadratic performance measure and system matrices depending on weight coefficients. The choice of these coefficients provides satisfaction of the state and mixed control-state constraints.

Keywords linear system · optimal control · state constraints · mixed constraints · the fundamental matrix · the matrix series

1 Introduction

In this article we consider a problem of control for linear dynamic system in which the main purpose of control is to keep the system in the state and mixed constraints under given deterministic perturbations with a minimum expenditure of energy. To solve this problem it is proposed to consider the auxiliary optimal control problem with quadratic performance measure with matrices depending on some weight coefficients. The choice of these coefficients eventually provides implementation of the state and mixed constraints under optimal control. We propose an approach to the construction of algorithmic support of the problem solution. This approach is based on a representation of the fundamental matrix in the form of an exponential matrix, arranged in a matrix row.
2 Problem statement

Consider the linear control system with constant coefficients
\[
\dot{x} = Ax(t) + Bu(t) + f(t), \quad x(0) = x^0,
\] (1)
where \(x\) – \(n\)-dimensional vector of state variables; \(u\) – \(r\)-dimensional control vector; \(A\) – constant \((n \times n)\)-matrix; \(B\) – constant \((n \times r)\)-matrix; \(f(t)\) – \(n\)-dimensional vector of external disturbances. The vector function \(f(t)\) is limited when \(t \geq 0\).

The system is controlled on a time interval \([0, T]\). In this case admissible controls \(u(t)\) are continuously differentiable functions.

Normal operation of the system involves at each time \(t \in [0, T]\) satisfaction of the state constraints
\[
x'(t)Q_1x(t) \leq 1, \quad (t = 1, 2, \ldots, s)
\] (2)
and mixed constraints
\[
x'(t)C_jx(t) + x'(t)P_ju(t) + u'(t)D_ju(t) \leq 1, \quad (j = 1, 2, \ldots, p),
\] (3)
where \(P_j\) – \((n \times r)\)-matrices; \(Q_1, C_j\) – non-negative definite, symmetric \((n \times n)\)-matrices; \(D_j\) – non-negative definite, symmetric \((r \times r)\)-matrices; bar ()' – here and subsequently the operation of transposition. Matrices \(P_j, C_j, D_j\) are such that matrices
\[
G_j = \left( \begin{array}{cc} C_j & \frac{1}{2}P_j \\ \frac{1}{2}P'_j & D_j \end{array} \right)
\]
are positive definite matrices.

It is required for a given perturbation \(f(t)\) to find such control \(u^*(t)\) that at each time \(t \in [0, T]\) the state (2) and mixed (3) constraints are satisfied on the trajectories of the system (1) and minimum of the following functional
\[
J(u) = \int_0^T u'(t)Ru(t)dt
\] (4)
is attained, where \(R\) – positive definite, symmetric \((r \times r)\)-matrix.

The functional (4) describes expenditure of energy.

Note that the need of considering this problem emerged from research connected with algorithmic support construction of problems arising in design automation of vibration isolation systems [1].

3 The auxiliary problem

Consider the optimal control problem
\[
\begin{align*}
\dot{x} &= Ax(t) + Bu(t) + f(t), \quad x(0) = x^0, \\
x(t), f(t) &\in E^n, u(t) \in E^r, t \in [0, T], \\
J(u(\cdot)) &= \frac{1}{2} \int_0^T (x'(t)Q(\alpha, \beta)x + x'P(\beta)u + u'R(\beta)u) dt \rightarrow \text{min},
\end{align*}
\] (5)
where $Q(\alpha, \beta)$, $P(\beta)$ and $R(\beta) - (n \times n)$, $(n \times r)$ and $(r \times r)$ appropriately matrices defined as follows

\begin{align*}
Q(\alpha, \beta) &= \alpha_1 Q_1 + \alpha_2 Q_2 + \ldots + \alpha_s Q_s + \beta_1 C_1 + \beta_2 C_2 + \ldots + \beta_p C_p, \\
P(\beta) &= \beta_1 P_1 + \beta_2 P_2 + \ldots + \beta_p C_p, \\
R(\beta) &= R + \beta_1 D_1 + \beta_2 D_2 + \ldots + \beta_p D_p.
\end{align*}

Here $\alpha_i \geq 0$ and $\beta_i \geq 0$ are weight coefficients.

Note the problem under consideration, as the problem of analytical design of optimal controller has been studied in detail in [2].

In the optimal control problem (5) by virtue of the fact that there is no constraints imposed on control from the maximum condition

\[ \frac{\partial H}{\partial u} = B' \psi - \frac{1}{2} P'(\beta)x - R(\beta)u = 0 \]

for Pontryagin function

\[ H = \psi' (Ax + Bu + f(t)) - \frac{1}{2} (x'Q(\alpha, \beta)x + x' P(\beta)u + u' R(\beta)u) \]

we get control:

\[ \hat{u} = R^{-1}(\beta) \left[ B' \psi - \frac{1}{2} P'(\beta)x \right]. \]  \hspace{1cm} (6)

Substituting the control (6) in the initial system (1) and in the adjoint system

\[ \dot{\psi} = -A' \psi + Q(\alpha, \beta)x + \frac{1}{2} P(\beta)u, \quad \psi(T) = 0, \]

combining them, we obtain the boundary value problem of the maximum principle:

\[ \begin{cases}
\dot{x} = Ax + BR^{-1}(\beta) \left[ B' \psi - \frac{1}{2} P'(\beta)x \right] + f(t), & x(0) = x^0; \\
\dot{\psi} = -A' \psi + Q(\alpha, \beta)x + \frac{1}{2} P(\beta)R^{-1}(\beta) \left[ B' \psi - \frac{1}{2} P'(\beta)x \right], & \psi(T) = 0.
\end{cases} \]  \hspace{1cm} (7)

Thus, if under certain weight coefficients we get the solution of the boundary value problem (7), the optimal control and optimal phase trajectory, respectively, can be written as

\[ \begin{cases}
u^*(\alpha, \beta, t) = R^{-1}(\beta) \left[ B' \psi(\alpha, \beta, t) - \frac{1}{2} P'(\beta)x(\alpha, \beta, t) \right], \\
x^*(\alpha, \beta, t) = x(\alpha, \beta, t),
\end{cases} \]  \hspace{1cm} (8)

where $x(\alpha, \beta, t)$ and $\psi(\alpha, \beta, t)$ is the solution of the boundary value problem (7).

In general, for solving boundary value problems there are no general methods of solution. The most fully investigated boundary-value problem for systems of linear differential equations (in particular the Abramov method), the system (7) is such system. But, nevertheless, it is of interest to develop new effective methods for solving linear boundary value problems. In our case, the interest is determined not only by the possibility to solve boundary value problems effectively, but to apply at algorithm development for solving the problem of choosing the weights $\alpha$, $\beta$, which provide performance of constraints (2) and (3), and at the same time provide minimum of the functional (4).
4 Boundary value problem

Consider the boundary value problem

\[
\begin{align*}
\dot{x} &= A_{11}x + A_{12}\psi + f(t), \quad x(0) = x^0, \\
\dot{\psi} &= A_{21}x + A_{22}\psi, \quad \psi(T) = 0,
\end{align*}
\]

(9)

wherein matrices \(A_{11}, A_{12}, A_{21}, A_{22}\) are defined as follows:

\[
A_{11} = A - \frac{1}{2}BR^{-1}(\beta)P'(\beta), \quad A_{12} = BR^{-1}(\beta)B',
\]

\[
A_{21} = Q(\alpha, \beta) - \frac{1}{4}P(\beta)R^{-1}(\beta)P'(\beta), \quad A_{22} = -A' + \frac{1}{2}P(\beta)R^{-1}(\beta)B'.
\]

Introducing the following notation:

\[
\tilde{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad y = \begin{pmatrix} x \\ \psi \end{pmatrix}, \quad \tilde{f}(t) = \begin{pmatrix} f(t) \\ 0 \end{pmatrix},
\]

we can write system (9) in the form

\[
\dot{y} = \tilde{A}y + \tilde{f}(t).
\]

(10)

The fundamental matrix \(F(t, \tau)\) of the homogeneous system, corresponding to the system (10), can be represented in a block form

\[
F(t, \tau) = \begin{pmatrix} F_{11}(t, \tau) & F_{12}(t, \tau) \\ F_{21}(t, \tau) & F_{22}(t, \tau) \end{pmatrix}.
\]

The solution of Cauchy problem for system (10) with the initial condition \(x(0) = x^0\), and some \(\psi(0) = \psi^0\) according to the Cauchy formula can be written as

\[
\begin{align*}
\begin{cases}
x(t) &= F_{11}(t, 0)x^0 + F_{12}(t, 0)\psi^0 + \int_0^t F_{11}(t, \tau)f(\tau)d\tau, \\
\psi(t) &= F_{21}(t, 0)x^0 + F_{22}(t, 0)\psi^0 + \int_0^t F_{21}(t, \tau)f(\tau)d\tau.
\end{cases}
\end{align*}
\]

(11)

Using the condition on the right end \(\psi(T) = 0\) from the second expression of (11) we get

\[
\psi(T) = F_{21}(T, 0)x^0 + F_{22}(T, 0)\psi^0 + \int_0^T F_{21}(T, \tau)f(\tau)d\tau = 0.
\]

We can define the initial condition \(\psi^0\):

\[
\psi^0 = -F_{22}^{-1}(T, 0) \left[ F_{21}(T, 0)x^0 + \int_0^T F_{21}(T, \tau)f(\tau)d\tau \right].
\]

(12)
Thus, if we have certain weight coefficients $\alpha, \beta$, we can write the solution of boundary value problem (9) as

\[
\begin{cases}
  x(\alpha, \beta, t) = F_{11}(t, 0)x^0 + \int_0^t F_{11}(t, \tau)f(\tau)d\tau + F_{12}(t, 0)\psi^0, \\
  \psi(\alpha, \beta, t) = F_{21}(t, 0)x^0 + \int_0^t F_{21}(t, \tau)f(\tau)d\tau + F_{22}(t, 0)\psi^0,
\end{cases}
\]

(13)

where $\psi^0$ is determined by expression (12).

For finding the initial condition $\psi^0$ (12) and the boundary problem solution $x(\alpha, \beta, t)$ and $\psi(\alpha, \beta, t)$ (13) it is necessary to calculate the fundamental matrix $F(t, \tau)$. To do this, we use the representation of the fundamental matrix in the form of a matrix exponential $F(t, \tau) = e^{\tilde{A}(t-\tau)}$, which in turn is represented in the form of a matrix series

\[
F(t, \tau) = e^{\tilde{A}(t-\tau)} = E + \tilde{A}(t-\tau) + \frac{1}{2!}\tilde{A}^2(t-\tau)^2 + \ldots + \frac{1}{k!}\tilde{A}^k(t-\tau)^k + \ldots
\]

(14)

Representation of the fundamental matrix in the form of a matrix series (14) can be used for the construction of algorithms for computing the fundamental matrix of any linear system of differential equations with constant coefficients [4] In accordance with the representation of the matrix $\tilde{A}$ from (14) we can get formulas to calculate the blocks of the matrix $F(t, \tau)$:

\[
\begin{align*}
F_{11}(t, \tau) &= E + \sum_{k=0}^{\infty} \frac{S_{11}^k(t-\tau)^{k+1}}{(k+1)!}, & F_{12}(t, \tau) &= \sum_{k=0}^{\infty} \frac{S_{12}^k(t-\tau)^{k+1}}{(k+1)!}, \\
F_{21}(t, \tau) &= \sum_{k=0}^{\infty} \frac{S_{21}^k(t-\tau)^{k+1}}{(k+1)!}, & F_{22}(t, \tau) &= E + \sum_{k=0}^{\infty} \frac{S_{22}^k(t-\tau)^{k+1}}{(k+1)!},
\end{align*}
\]

(15)

where $(n \times n)$ - matrices $S_{11}^k, S_{12}^k, S_{21}^k, S_{11}^k$ are defined by the following recurrence relations

\[
\begin{align*}
\begin{pmatrix}
  S_{11}^0 & S_{12}^0 \\
  S_{21}^0 & S_{22}^0
\end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\
  A_{21} & A_{22} \end{pmatrix}, \\
\begin{pmatrix}
  S_{11}^{k+1} & S_{12}^{k+1} \\
  S_{21}^{k+1} & S_{22}^{k+1}
\end{pmatrix} &= \begin{pmatrix} S_{11}^kA_{11} + S_{12}^kA_{21} & S_{11}^kA_{12} + S_{12}^kA_{22} \\
  S_{21}^kA_{11} + S_{22}^kA_{21} & S_{21}^kA_{12} + S_{22}^kA_{22} \end{pmatrix},
\end{align*}
\]

(16)

For an approximate calculation of the blocks of matrix $F(t, \tau)$, according to (16) sum of the series will be limited to a finite number of terms of the series, when the addition of a new member of the series changes each element of the partial sum less than the required accuracy.
5 The problem of searching the optimal weight coefficients

Consider the problem of finding the weight coefficients $\alpha^* > 0$, $\beta^* > 0$ satisfying inequalities

$$
\begin{cases}
x'^*(\alpha, \beta, t)Q_i x^*(\alpha, \beta, t) \leq 1, \\
x'^*(\alpha, \beta, t)C_j x^*(\alpha, \beta, t) + x'^*(\alpha, \beta, t)P_j u^*(\alpha, \beta, t) + \\
+ u'^*(\alpha, \beta, t)D_j u^*(\alpha, \beta, t) \leq 1 \\
(t = 1, \ldots, s), \quad (j = 1, \ldots, p), \quad t \in [0, T],
\end{cases}
$$

and minimizing function

$$
\varphi(\alpha, \beta) = J(u^*(\alpha, \beta, \cdot)) = \int_0^T u'^*(\alpha, \beta, t)Ru^*(\alpha, \beta, t)dt,
$$

where $x^*(\alpha, \beta, t)$ and $u^*(\alpha, \beta, t)$ respectively be optimal trajectory and optimal control in the auxiliary problem (5), defined by certain weight coefficients $\alpha, \beta$.

For solving this problem it is offered to divide closed interval $[0, T]$ into intervals with points $t_0 = 0$, $t_1, \ldots, t_{N-1}, t_N = T$ and consider the following approximating problem: find weight coefficients $\alpha^* > 0$, $\beta^* > 0$, that minimize function (18) on the set defined by inequalities

$$
\begin{cases}
x'^*(\alpha, \beta, t_k)Q_i x^*(\alpha, \beta, t_k) \leq 1, \\
x'^*(\alpha, \beta, t_k)C_j x^*(\alpha, \beta, t_k) + x'^*(\alpha, \beta, t_k)P_j u^*(\alpha, \beta, t_k) + \\
+ u'^*(\alpha, \beta, t_k)D_j u^*(\alpha, \beta, t_k) \leq 1 \\
(i = 1, 2, \ldots, s), \quad (j = 1, 2, \ldots, p),
\end{cases}
$$

where $k = 1, 2, \ldots, N$.

**Remark 1.** For the construction of specific algorithms in calculating the objective function (18) it is possible to represent integral in the form of simple cubature formulas (rectangle, trapezoidal, Simpson methods), corresponding to the above partition of closed interval $[0, T]$. For solving resulting problem of mathematical programming are being developed special problem-oriented iterative procedures that take account of the specifics of concrete disturbances $f(t)$.

In implementing the iterative procedures it is necessary to choose an initial approximation on the set of admissible weights. Here admissible weights are weights $\alpha, \beta$ that satisfy inequalities (17)

Let found in certain weight coefficients $\alpha^0, \beta^0$ solution of boundary value problem (7) $x(\alpha^0, \beta^0, t)$ and $\psi(\alpha^0, \beta^0, t)$, and in accordance with (8) the solution of auxiliary optimal control problem (5). If the optimal control $u^*(\alpha^0, \beta^0, t)$ and optimal trajectory $x^*(\alpha^0, \beta^0, t)$, respectively, satisfy the constraints (2) and (3), it can be taken $\alpha^0, \beta^0$ as an initial approximation. Otherwise, we consider the problem of choosing weights which provide performance of inequalities

$$
\varphi_l(\gamma) \leq 1, \quad (l = 1, 2, \ldots, (s + p)N),
$$

where $\varphi_l(\gamma)$ is function describing $l$-inequality in (19); $\gamma = (\alpha, \beta)$. 

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Suppose in certain $\gamma^0 = (\alpha^0, \beta^0)$ some of inequalities (19) are violated and herewith are satisfied

$$\varphi_l(\gamma^0) < 1 - \delta \quad \text{for} \ l \in I \quad \text{and} \quad \varphi_l(\gamma^0) \geq 1 - \delta \quad \text{for} \ l \in J,$$

where $\delta$ is small positive number; $I$ and $J$ are indexing sets, for which the corresponding constraints are respectively satisfied or not satisfied. Obviously $I \cup J = 1, 2, \ldots (s + p)N$.

For searching weights which provide fulfillment of the inequalities (19) is constructed an iterative process of falling into the region defined by the inequalities (19). Without discussing the details which take place in a particular implementation, we present the procedure for the implementation of one step of the iterative process.

1. Descent direction $h$ is constructed as follows

$$h = - \sum_{l \in J} \lambda_l \nabla \varphi_l(\gamma^0),$$

where $\nabla \varphi_l(\gamma^0)$ is a gradient of the function $\varphi_l(\gamma)$, evaluated at $\gamma^0$; $\lambda_l$ are non-negative integers. The coefficients $\lambda_l$ are chosen so that the inequalities

$$\nabla \varphi_l(\gamma^0), h = - \left( \nabla \varphi_l(\gamma^0), \sum_{l \in J} \lambda_l \nabla \varphi_l(\gamma^0) \right) = \sum_{l \in J} \lambda_l (\nabla \varphi_l(\gamma^0), \nabla \varphi_l(\gamma^0)),$$

are satisfied for all indices $l$ at which inequalities (19) are violated.

Note that in this case the direction of descent (21) determines the direction of decreasing at the point $\gamma^0$ of all functions wherein inequalities (19) are violated.

2. The next point of the iterative sequence we define by making a step in the direction of $h$

$$\gamma^1 = \gamma^0 + \chi h.$$

The step size $\chi > 0$ is chosen so that to fulfill the condition

$$\max_{l \in J} \varphi_l(\gamma^1) < \max_{l \in J} \varphi_l(\gamma^0)$$

in conditions:

$$\varphi_l(\gamma^0) < 1 \quad \text{for all} \ l \in I$$

6 To the calculation of the partial derivatives with respect to weights of the boundary-value problem solution

Note that in implementing iterative procedures it is necessary to calculate the partial derivatives of the boundary-value problem solution (9) $x(\gamma, t)$ and $\psi(\gamma, t)$ with respect to weights $\gamma_l$.

In accordance with (13) the solution of (9), given that the fundamental matrix depends on the choice of weights can be written as

$$y(\gamma, t) = F(\gamma, t, 0)y^0 + \int_0^t F(\gamma, t, \tau)\tilde{f}(\tau)d\tau,$$
where
\[ y(\gamma, t) = \left( \begin{array}{c} x(\gamma, t) \\ \psi(\gamma, t) \end{array} \right), \quad y^0 = \left( \begin{array}{c} x^0 \\ \psi^0 \end{array} \right), \quad F(\gamma, t, \tau) = \left( \begin{array}{cc} F_{11}(\gamma, t, \tau) & F_{12}(\gamma, t, \tau) \\ F_{21}(\gamma, t, \tau) & F_{22}(\gamma, t, \tau) \end{array} \right). \]

Because of the representation (23) calculation of the partial derivatives of the function \( y(\gamma, t) \) with respect to weights \( \gamma_i \) is reduced to finding the partial derivatives of the fundamental matrix \( F(\gamma, t, \tau) \) with respect to weights \( \gamma_i \). To calculate the fundamental matrix has been used above its representation in the form of matrix series
\[ F(\gamma, t, \tau) = e^{-A(\gamma)(t-\tau)} = E + \sum_{k=1}^{\infty} \frac{1}{k!} A^k(\gamma)(t-\tau)^k. \quad (24) \]

At fixed \( t \) and \( \tau \) the series (24) is functional matrix series of \( \gamma \) that converges for any value \( \gamma \). Differentiating on \( \gamma \), this series, we obtain the matrix series consisting of derivatives
\[ \sum_{k=1}^{\infty} \frac{1}{k!} (t-\tau)^k \left[ \sum_{s=0}^{k-1} \tilde{A}^s(\gamma) \frac{\partial A}{\partial \gamma_s} A^{k-s-1}(\gamma) \right]. \quad (25) \]

**Theorem 1.** For fixed values \( t \) and \( \tau \) functional matrix series of \( \gamma \) (27) composed of derivatives of (26) converges uniformly and, therefore, for the partial derivatives of the fundamental matrix \( F(\gamma, t, \tau) \) with respect to weights \( \gamma_i \) the (26) holds
\[ \frac{\partial F(\gamma, t, \tau)}{\partial \gamma_i} = \sum_{k=1}^{\infty} \frac{1}{k!} (t-\tau)^k \left[ \sum_{s=0}^{k-1} \tilde{A}^s(\gamma) \frac{\partial A}{\partial \gamma_i} A^{k-s-1}(\gamma) \right]. \quad (26) \]

**Proof.** For the convergence of matrix series it is sufficient that the series consisting of terms of sequences converges [5].

Consider the functional series of the variable \( \gamma \), consisting of norms of the terms of the series (25)
\[ \sum_{k=1}^{\infty} \frac{1}{k!} |t-\tau|^k \left\| \sum_{s=0}^{k-1} \tilde{A}^s(\gamma) \frac{\partial A}{\partial \gamma_s} A^{k-s-1}(\gamma) \right\|. \quad (27) \]

We have
\[ \frac{1}{k!} |t-\tau|^k \left\| \sum_{s=0}^{k-1} \tilde{A}^s(\gamma) \frac{\partial A}{\partial \gamma_s} A^{k-s-1}(\gamma) \right\| \leq |t-\tau|^k \left\| \frac{\partial A}{\partial \gamma_i} (\gamma) \right\| \frac{1}{(k-1)!} \| A(\gamma) \|^{k-1}. \]

Thus for series (27) there is a majorant
\[ \sum_{k=1}^{\infty} |t-\tau|^k \left\| \frac{\partial A}{\partial \gamma_i} (\gamma) \right\| \frac{1}{(k-1)!} \| A(\gamma) \|^{k-1}. \]

This series is uniformly convergent. In fact, we represent it in the form of
\[ \sum_{k=1}^{\infty} |t-\tau|^k \left\| \frac{\partial A}{\partial \gamma_i} (\gamma) \right\| \frac{1}{(k-1)!} \| A(\gamma) \|^{k-1} = |t-\tau|^k \left\| \frac{\partial A}{\partial \gamma_i} (\gamma) \right\| \left( 1 + \sum_{m=1}^{\infty} \frac{\| A(\gamma) \|^m}{m!} |t-\tau|^m \right). \]
Expression
\[
1 + \sum_{m=1}^{\infty} \frac{\|A(\gamma)\|^m}{m!} |t - \tau|^m
\]
is expansion in exponential series \(e^{\|A(\gamma)\|(t-\tau)}\) that is uniformly convergent.

It follows that functional series (27) of \(\gamma\) is majorized by uniformly convergent series and therefore it is uniformly convergent. Thus the matrix series (25) consisting of derivatives of (24) is uniformly convergent, which proves (26). The theorem is proved.

**Remark 2.** In developing algorithmic support of the procedure for selecting weights for the approximate calculation of the partial derivatives of functions \(y(\gamma, t)\) with respect to weights used in the calculation of partial derivatives of the fundamental matrix \(F(\gamma, t, \tau)\) with respect to weights \(\gamma_i\), their representation in the form of matrix series (26).

7 **Conclusion**

In this work we propose an approach to the construction of algorithmic support of problem solution. This problem is an optimal control problem for linear dynamic system under deterministic given perturbations in which the main purpose of control is to keep the system in the state and mixed constraints with a minimum expenditure of energy.

For solving this problem we propose to consider the auxiliary problem of optimal control with quadratic performance criteria, which matrices depend on some weights. Choice of these weights provides satisfaction of the state and mixed constraints with a minimum initial performance criteria.

Algorithmic support construction of boundary value problem solution and of iterative procedure for selecting the optimal weights is based on the representation of the fundamental matrix in the form of exponential matrix, arranged in a matrix series.

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**References**

THE FIXED POINT METHOD IN POLYNOMIAL WITH RESPECT TO
STATE PROBLEM OF PARAMETRIC OPTIMIZATION

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Abstract. A new approach for improving control parameters of polynomial with respect to state systems is proposed. The approach is based on the construction and solution of the problem of fixed point defined by the projection operator. The procedure can improve controls satisfying maximum principle.

Keywords: parametric optimization, fixed point problem

1 Polynomial with respect to state problem of parametric optimization

We consider the problem of optimizing a dynamic system by control parameters

\[
\Phi(u) = \varphi(x(t)) + \int_T F(x(t), u, t) dt \to \min_{u \in U}
\]

\[
\dot{x}(t) = f(x(t), u, t), \quad x(t_0) = x^0, \quad u \in U, \quad t \in T = [t_0, t_1],
\]

where \( u = (u_1, \ldots, u_m) \) - vector of control parameters (control) with values in a compact set \( U \subseteq \mathbb{R}^m \). Functions \( f(x, u, t) \), \( F(x, u, t) \) are polynomials of an entire degree \( k \geq 1 \) with respect to \( x \) with coefficients, depending continuously on \( u, t \), on the set \( \mathbb{R}^n \times U \times T \), function \( \varphi(x) \) - is a polynomial of degree \( k \) on \( \mathbb{R}^n \). The initial state \( x^0 \) and the interval \( T \) are fixed.

Let us form the Pontryagin function

\[
H(\psi, x, u, t) = \langle \psi, f(x, w, t) \rangle - F(x, w, t).
\]

Consider a vector standard adjoint system

\[
\dot{\psi}(t) = -H_x(\psi(t), x(t), w, t), \quad t \in T
\]

and introduce a vector modified adjoint system

\[
\dot{p}(t) = -H_x(p(t), x(t), w, t) - \frac{1}{2!} \langle H_x(p(t), x(t), w, t), z_x \rangle -
\]

\[
\ldots - \frac{1}{k!} \langle \ldots \langle H_x(p(t), x(t), w, t), z \rangle_x, z \rangle_x, \ldots, z \rangle_x.
\]

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For admissible controls $u$, $v$ let us denote $Δ_vΦ(u)$ as an increment of the objective function; $x(t,v)$, $t ∈ T$ - is a solution of system $(2)$ at $w = v$, $x(t_0,v) = x^0$; $ψ(t,v)$, $t ∈ T$ - is a solution of system $(3)$ at $w = v$, $x = x(t,v)$, $ψ(t_1,v) = −φ_x(x(t_1,v))$; $p(t,u,v)$, $t ∈ T$ - is a solution of system $(4)$ at $w = u$, $x(t) = x(t,u)$, $z = x(t,v) − x(t,u)$ with a boundary condition

$$p(t_1,u,v) = −φ_x(x(t_1,u)) − \frac{1}{2!}⟨φ_x(x(t_1,u)),y⟩ x − \ldots − \frac{1}{k!}⟨\ldots⟨φ_x(x(t_1,u)),y⟩ x, y⟩ x \ldots, y⟩ x,$$

where partial derivatives with respect to $x$ are calculated at $y = x(t_1,v) − x(t_1,u)$. It is evident that $p(t,u,u) = ψ(t,u)$, $t ∈ T$.

Let us set the improvement control problem for $u ∈ U$: to find a control $v ∈ U$ satisfying the condition $Δ_vΦ(u) ≤ 0$.

Control vector of parameters in the problem $(1)$, $(2)$ is considered as a constant vector function of time on the interval $T$. Then from the exact formulas of increment of objective functional in the optimal control problem $[1]$ as an obvious corollary we obtain the first and the second exact formulas of increment of objective function accordingly in the problem $(1)$, $(2)$.

$$Δ_vΦ(u) = − \int_T Δ_vH(p(t,u,v),x(t,v),u,t)dt, \tag{5}$$

$$Δ_vΦ(u) = − \int_T Δ_vH(p(t,v,u),x(t,u),v,t)dt. \tag{6}$$

Let us introduce a mapping using the formula

$$W^*(u,v) = \arg\max_{w ∈ U} \int_T H(p(t,u,v),x(t,v),w,t)dt, \ u ∈ U, \ v ∈ U. \tag{7}$$

According to the formula $(5)$ for optimality of control $u ∈ U$ it is sufficient (and necessary) that

$$\int_T Δ_vH(p(t,u,v),x(t,v),u,t)dt ≤ 0, \ v ∈ U.$$

To fulfill the last inequality it is sufficient to require a fulfillment of the condition

$$u = W^*(u,v), \ v ∈ U \tag{8}$$

The sufficient optimality condition, based on the formula $(6)$, accordingly has the form

$$u = W^*(v,u), \ v ∈ U \tag{9}$$

In case of differentiability of the function $H$ with respect to $u$ and convexity of the set $U$ the necessary optimality condition for the control $u ∈ U$ in the problem $(1)$, $(2)$ has the form of known differential maximum principle (DPM) $[2]$

$$u = \arg\max_{w ∈ U} \int_T (H_u(ψ(t,u),x(t,u),u,t),w)dt. \tag{10}$$

Let us extract the condition

$$u = W^*(u,u), \tag{11}$$
which is obtained from sufficient conditions (8), (9) at \( v = u \). It is easy to show that (10) is a corollary of (11).

Consider an important for applications subclass of linear with respect to control problems

\[
\Phi(u) = \varphi(x(t_1)) + \int_T ((a(x(t), t), u) + d(x(t), t))dt \rightarrow \min_{u \in U},
\]

\[
\dot{x}(t) = A(x(t), t)u + b(x(t), t), \quad x(t_0) = x^0, \quad u \in U, \quad t \in T,
\]

where the matrix function \( A(x, t) \), vector functions \( a(x, t) \), \( b(x, t) \), functions \( \varphi(x) \), \( d(x, t) \) are polynomial with respect to \( x \) and continuous with respect to \( t \) on the set \( \mathbb{R}^n \times T \). The set \( U \in \mathbb{R}^m \) is convex and compact.

In problem (12), (13) the differential maximum principle for control \( u \in U \) is represented as (11). The Pontryagin function, the exact formulas of the increment of objective function (5), (6) and the mapping \( W^*(u, v) \) have the form

\[
H(\psi, x, u, t) = H_0(\psi, x, t) + \langle H_1(\psi, x, t), u \rangle,
\]

\[
H_0(\psi, x, t) = \langle \psi, b(x, t) \rangle - d(x, t), \quad H_1(\psi, x, t) = A^T(x, t)\psi - a(x, t),
\]

\[
\Delta_v \Phi(u) = -\langle \int_T H_1(p(t, u, v), x(t, v), t)dt, v - u \rangle,
\]

\[
\Delta_v \Phi(u) = -\langle \int_T H_1(p(t, u, v), x(t, v), t)dt, v - u \rangle,
\]

\[
W^*(u, v) = \operatorname{argmax}_{w \in U} \langle \int_T H_1(p(t, u, v), x(t, v), t)dt, w \rangle, \quad u \in U, \quad v \in U.
\]

For admissible control \( u \in U \) let us define the projection \( u^\alpha \), \( \alpha > 0 \) using the relation

\[
u^\alpha = P_U(u + \alpha \int_T H_1(\psi(t, u), x(t, u), t)dt),
\]

where \( P_U \) is a projection operator on the set \( U \) in the Euclidean norm.

Differential maximum principle in problem (12), (13) for control \( u \in U \) on the basis of projection property (16) is represented in the form

\[
u = u^\alpha, \quad \alpha > 0.
\]

Note, that to fulfill (11) it is sufficient to verify the condition (14) at least for one \( \alpha > 0 \). Conversely, from the condition (11) it follows that (14) is fulfilled for all \( \alpha > 0 \).

### 2 Fixed point methods in parametric optimization

The exact formulas (5), (6) enable us to construct methods of fixed point in problem (1), (2) using the operation for the maximum (7).

Let us consider the improvement control problem for \( u^0 \in U \).

**The first fixed point method**: for given \( u^0 \in U \) we define the mapping \( W_1^* \) using the relation \( W_1^*(v) = W^*(u^0, v) \), \( v \in U \) and find a solution \( v = v(u^0) \) of a fixed point problem

\[
v = W_1^*(v).
\]
As \( W_1^*(v) \in U \), then the obtained control \( v = v(u^0) \) is admissible \((v \in U)\). By virtue of definition of the mapping \( W_1^* \) we obtain

\[
\int_T H(p(t, u^0, v), x(t, v), v, t)dt \geq \int_T H(p(t, u^0, v), x(t, v), u^0, t)dt.
\]

From here and from the formula (5) follows \( \Delta_v\Phi(u^0) \leq 0 \).

The second fixed point method: for given \( u^0 \in U \) define the mapping \( W_2^* \) using the relation \( W_2^*(v) = W^*(v, u^0) \), \( v \in U \) and find a solution of a fixed point problem.

\[
v = W_2^*(v).
\]

By analogy with the first procedure (15), we find that the solution of equation (16) in the second method is admissible and provides an improvement.

Thus, the fixed point methods consist in search for fixed points of corresponding mappings \( W_1^* \) and \( W_2^* \).

Let us consider the sets of fixed points \( V_1(u^0) = \{ v \in U : v = W_1^*(v) \} \) and \( V_2(u^0) = \{ v \in U : v = W_2^*(v) \} \) in corresponding procedures of improvement. If \( u^0 \in V_1(u^0) \), then \( u^0 \) satisfies condition (11). Conversely, if \( u^0 \) satisfies condition (11), then \( u^0 \) is a solution of equation (15), that is \( u^0 \in V_1(u^0) \). Hence, the condition (11) for \( u^0 \in U \) characterizes inclusion \( u^0 \in V_1(u^0) \). Similarly the condition (11) is equivalent to inclusion \( u^0 \in V_2(u^0) \).

Thus, the following statement is true.

Lemma 1. Control \( u^0 \in U \) satisfies the condition (11) then and only then, when \( u^0 \in V_1(u^0) \) (respectively \( u^0 \in V_2(u^0) \)).

Corollary 1. In problem (12), (13) control \( u^0 \in U \) satisfies the differential maximum principle then and only then, when \( u^0 \in V_1(u^0) \) (respectively \( u^0 \in V_2(u^0) \)).

Thus, the absence of fixed points in methods indicates a non-optimality of control \( u^0 \in U \) in problem (12), (13).

Property of non-uniqueness of fixed points in methods is a necessary condition for strict improvement of controls, satisfying the DMP in the problem (12), (13).

Indicate the conditions, under which there is a strict improvement \( \Phi(v) - \Phi(u^0) \) at \( v \in V_1(u^0) \) (\( v \in V_2(u^0) \)). Restrict ourselves to the case of a linear with respect to control problem (12), (13).

Define the vector-functions in the first and second methods respectively in the form

\[
g_1(v) = \int_T H_1(p(t, u^0, v), x(t, v), t)dt, \quad v \in U,
\]

\[
g_2(v) = \int_T H_1(p(t, v, u^0), x(t, u^0), t)dt, \quad v \in U.
\]

Herewith mappings \( W_1^* \) and \( W_2^* \) in methods of fixed points take respectively the form

\[
W_1^*(v) = \arg\max_{w \in U} \langle g_1(v), w \rangle, \quad v \in U,
\]

\[
W_2^*(v) = \arg\max_{w \in U} \langle g_2(v), w \rangle, \quad v \in U.
\]

The first (14) and the second (15) formulas of the increment of objective function at controls \( u^0 \in U, \quad v \in U \) are written in the form

\[
\Delta_v\Phi(u^0) = -\langle g_1(v), v - u^0 \rangle,
\]
\[ \Delta_v \Phi(u^0) = -\langle g_2(v), v - u^0 \rangle. \]

Assume that \( v \in U \) - is a fixed point in method. Then from the formulas of increment follows that the strict improvement is guaranteed (including for control \( u^0 \), satisfying DMP), if vectors \( v - u^0 \) and \( g_1(v) \) (respectively \( g_2(v) \)) are not orthogonal. In particular, for scalar control \((m = 1)\) strict improvement is guaranteed, if \( v \) is not equal to \( u^0 \) and function \( g_1 \) (respectively \( g_2 \)) is not equal to zero at the point \( v \).

The points \( v \in U \), at which function \( g_1(g_2) \) is equal to zero, obviously are fixed points of mapping \( W_1^*(W_2^*) \). Let us call these fixed points singular, the others - nonsingular.

The points \( v \in U \), for scalar control \((m = 1)\), strict improvement is guaranteed, if \( v \) is not equal to \( u^0 \) and function \( g_1 \) (respectively \( g_2 \)) is not equal to zero at the point \( v \).

The proposed methods indicate the possibility in principle of realization of nonlocal improvement of control parameters in considered class of polynomial with respect to state problems. The complexity of construction of vector of control parameters that improves is determined by the complexity of solving the problem of fixed point of auxiliary vector-functions in methods. The mapping \((7)\) is generally determined by many-valued and discontinuous vector functions \( W_1^* \) and \( W_2^* \), values of which are calculated through the operation of integration of phase and adjoint systems and the operation for the maximum. Compensation for the search of fixed points of the mapping is a property of nonlocality of improvement and a possibility of strict improvement of controls that satisfy differential maximum principle.

### 3 Examples

Let us illustrate the work of the proposed fixed point methods on simple examples.

**Example 1** (improvement of control).

\[ \Phi(u) = \frac{1}{2} \int_0^2 x^2(t) dt \rightarrow \min, \]

\[ \dot{x}(t) = u, \quad x(0) = 1, \quad u \in U = [-1, 1], \quad t \in T = [0, 2]. \]

In this case \( H = pu - \frac{1}{2}x^2 \), the modified adjoint system \( \dot{p}(t) = x(t) + \frac{1}{2}y(t), \quad p(2) = 0 \).

Consider control \( u^0 = 0 \) with a corresponding phase trajectory \( x(t, u^0) = 1 \), \( t \in T \) and a value of functional \( \Phi(u^0) = 1 \). Let us set a problem of improvement of control \( u^0 \).

Apply the first fixed point method. We have

\[ x(t, v) = vt + 1, \quad p(t, u^0, v) = t + \frac{vt^2}{4} - (v + 2), \quad t \in T, \]

\[ W_1^*(v) = \text{sign}g_1(v), \quad g_1(v) = \int_0^2 p(t, u^0, v) dt = -2 - \frac{4}{3}v, \quad v \in U. \]
A unique zero of the switching function \( g_1 \) is the point \( v = -\frac{3}{2} \notin [-1, 1] \). Consequently, there are no singular fixed points. Using the method of substitution a unique nonsingular fixed point \( v = -1 \) is defined.

Let us discuss the procedure. Since \( u^0 = 0 \) does not satisfy the differential maximum principle. Since \( v = -1 \neq u^0 \) and \( g_1(-1) \neq 0 \), then \( v = -1 \) strictly improves \( u^0 = 0 \) with a value of the increment \( \Delta_v \Phi(u^0) = -(v - u^0)g_1(v) = -\frac{2}{3} \).

**Example 2** (improvement of control, satisfying the maximum principle).

\[
\Phi(u) = \frac{1}{2} \int_0^1 u x^2(t) dt \to \min_{u \in U},
\]

\[
\dot{x}(t) = u, \ x(0) = 0, \ u \in U = [-1, 1], \ t \in T = [0, 1].
\]

Let us set a problem of improvement of control \( u^0 = 0 \) with a corresponding phase trajectory \( x(t, u^0) = 0 \), \( t \in T \) and a value of the objective function \( \Phi(u^0) = 0 \).

Pontryagin function is \( H = \psi u - \frac{1}{2} u x^2 \), standard adjoint system is \( \dot{\psi}(t) = u x(t), \ \psi(1) = 0 \). We have \( \psi(t, u^0) = 0, \ t \in T \). From this we obtain \( H_u(\psi(t, u^0), x(t, u^0), u^0, t) = 0, \ t \in T \).

Thus, \( u^0 \) is a special control, i.e. it satisfies the differential maximum principle with degeneracy.

Apply the first method of fixed point. Define a modified adjoint system \( \dot{p}(t) = u x(t) + \frac{1}{2} u y(t), \ p(1) = 0 \); the mapping \( W_1^*(v) = \text{sign} \int_0^1 (p(t, u^0, v) - \frac{1}{2} x^2(t, v)) dt, \ v \in U \). We obtain \( p(t, u^0, v) = 0, \ x(t, v) = vt, \ t \in T, \ W_1^*(v) = \text{sign}(-\frac{1}{6} v^2), \ v \in U \).

A unique zero of the switching function \( g_1(v) = -\frac{1}{6} v^2 \) is the point \( v = 0 \in U \), which is the singular fixed point of the mapping \( W_1^* \). Nonsingular fixed point \( v = -1 \) strictly improves \( u^0 = 0 \) with a value of the increment \( \Delta_v \Phi(u^0) = -(v - u^0)g_1(v) = -\frac{1}{6} \).

This example demonstrates the following properties of constructed methods:
- control, satisfying condition (11), is a fixed point of mapping in the methods;
- possibility of nonunique solution of the problem of fixed point in the methods;
- possibility of strict improvement of control that satisfies the differential maximum principle.

### 4 Conclusion

Let us distinguish the following properties of methods of fixed points that favourably distinguish them from the gradient methods.

1. Nonlocality of improvement of control parameters without procedure of variation on small parameter.

2. Possibility of strict improvement of control parameters that satisfy differential maximum principle.

In linear control problem such a possibility is conditioned by non-uniqueness of solution of the problem of fixed point in methods.

3. Possibility of improvement of control parameters on nonconvex sets in terms of absence of the property of differentiability on control for functions \( f(x, u, t), \ F(x, u, t) \).
Gradient methods do not provide such a possibility.

In the case where in procedures of fixed point there are no admissible fixed points the method does not work and we have to turn to other methods of improvement. In problem (12), (13) this means that the improvable vector of control parameters does not satisfy differential maximim principle and hence, is not optimal.

**Literature**


A Mathematical Model of Marketing Advertisement for Insurance Companies

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Abstract

We examine optimal advertising spending in a duopolistic market where each firm's market share depends on its own and its competitor's advertising decisions. We extend a differential game theory model of advertising which is based on the classic models such as Vidale-Wolfe, Sethi, Sorger and Prasad advertising models by incorporating profit functions obtained due to differences of exchange currencies. We implement our model for Mongolian insurance companies. Numerical results show the efficiency of our proposed model.

Key Words and Phrases: Stochastic optimal control, competitive strategy, Hamilton-Jacobi-Bellman equation, GARCH model.

1 Introduction

We examine a mathematical model of Marketing advertisement for Mongolian insurance companies. The study has three main objectives. Firstly, the exact setting of advertising spending will be defined for a business environment of uncertainty. Secondly, in its industry should develop its advertising model and determine a structure its market. Thirdly, a differential game theory as a model for competitive players will be studied.

Stochastic Optimal Control Theory

The following stochastic optimal control problem is considered for a dynamic system with the state vector \( x(t) \in \mathbb{R}^n \), the admissible control vector \( u(t) \in U \subseteq \mathbb{R}^m \) (where \( U \) is a time-invariant, convex, and closed subset of \( \mathbb{R}^m \)), and the standard vector Brownian motion \( W(t) \in \mathbb{R}^k \). For the dynamic system described by the stochastic differential equation \( dx(t) = \)
\( f(x(t), u(t), t)dt + g(x(t), u(t), t)dW(t) \) with the given deterministic initial state \( x_0 \) at the fixed initial time \( t_0 \), \( x(t_0) = x^0 \) find a piecewise continuous control vector \( u(t) \in U \) for all times \( t \) in the fixed time interval \( t \in [t_0, T] \), such that the objective function

\[
J(x, t) = E \left[ \int_{t_0}^{T} F(x(t), u(t), t)dt + \varphi(x(T)) \right]
\] (1.1)

is maximized.

**Theorem 1.** *(Hamilton-Jacobi-Bellman theorem).* If the partial differential equation

\[
-\mathcal{J}_t(x, t) = \max_{u \in U} \left\{ F(x, u) + \frac{\partial \mathcal{J}(x, t)}{\partial x} f(x, u, t) + \frac{1}{2} \text{tr} \left( \frac{\partial^2 \mathcal{J}(x, u, t)}{\partial x^2} g(x, u, t) g^T(x, u, t) \right) \right\}
\]

with the boundary condition \( \mathcal{J}(x, T) = \varphi(x) \) admits a unique solution, the globally optimal state feedback control law is

\[
u(x) = \max_{u \in U} \left\{ F(x, u) + \frac{\partial \mathcal{J}(x, t)}{\partial x} f(x, u, t) + \frac{1}{2} \text{tr} \left( \frac{\partial^2 \mathcal{J}(x, u, t)}{\partial x^2} g(x, u, t) g^T(x, u, t) \right) \right\}
\]

A rigorous proof of this theorem can be found in Yong-Zhou [39].

**Theorem 2.** *(Standard Brownian motion, Breiman [6]).* A Brownian motion is called standard if

\[
E(dW) = 0, \ dW \ dt = 0, \ dt dW = 0
\]

**Dynamic advertising model (Vidale-Wolfe, 1957)**

\[
J(x) = \max_{u > 0} \left[ \int_{t_0}^{T} e^{-rt}(mx - u)dt \right]
\] (1.2)

\[
\begin{cases}
\dot{x} = \rho u (1 - x) - \delta x \\
x(t_0) = x^0 \\
x(T) = x^T
\end{cases},
\]

(1.3)

where \( x \)-market share, \( u \)-control variable, \( J \)-value function, \( r \)-discount rate, \( \delta \)-loss rate of market, \( \rho \)-advertising effectiveness, \( m \)-profit margin on sales, \( T \)-planning time.
Stochastic advertising model (Sethi, 1983)

\[
\mathcal{J}(x(t)) = \max_{u(t) \geq 0} E \left[ \int_0^\infty e^{-rt}(mx(t) - u(t)^2)dt \right],
\]

where \(E\)-expectation operator. An advertising model is determined by the following equation of state of dynamic model.

\[
\frac{dx(t)}{dt} = \left( pu(t)\sqrt{1-x(t)} - \delta x(t) \right)
\]

The above equation includes a risk of volatility (\(\sigma\)) and Brownian process \(dW\).

\[
\mathcal{J}(x(t)) = \max_{u(t) \geq 0} E \left[ \int_0^\infty e^{-rt}(mx(t) - u(t)^2)dt \right]
\]

\[
\left\{ \begin{array}{l}
\frac{dx}{dt} = \left( pu(t)\sqrt{1-x(t)} - \delta x(t) \right) dt + \sigma(x)dW \\
x(0) = x_0 \\
u(t) \geq 0 \\
\delta(x) \geq 0, \ x \in (0, 1) \\
u(x) \geq 0, \ x \in (0, 1]
\end{array} \right.
\]

Competition advertising model (Sorger, 1989)

Dynamic advertising competition among duopolists battling for market share has been investigated by, among others, Deal [8], Sethi and Thompson [29], Erickson [10], Chintagunta and Vilcassim [7], and surveyed by Jorgensen [17]. These studies have used the Lanchester model of combat to characterize the market share evolution over time for the competing firms. (Kimball [18], Little [19]). Although there is no consensus on the competitive extension of the Lanchester model of combat is

\[
\left\{ \begin{array}{l}
\frac{dx}{dt} = \rho_1 u_1(x, y)(1-x) - \rho_2 u_2(x, y)x \\
\frac{dy}{dt} = \rho_2 u_2(x, y)x - \rho_1 u_1(x, y)(1-x)
\end{array} \right.
\]

where \(x(t)\) and \(y(t)\) represent the market shares of the two firms, whose parameters and decision variables are indexed 1 and 2 respectively. Note that \(x(t) + y(t) = 1\). A related extension is due to Sorger [32]. He uses a special case of the Lanchester model to take advantage of Sethis [30] formulation that results in an explicit solution. This is,

\[
\mathcal{J}_1(x) = \max_{u_1 \geq 0} \left[ \int_0^\infty e^{-r_1 t}(m_1 x - c_1 u_1^2)dt \right]
\]

\[
\mathcal{J}_2(y) = \max_{u_2 \geq 0} \left[ \int_0^\infty e^{-r_2 t}(m_2 y - c_2 u_2^2)dt \right]
\]
\begin{align}
\left\{ \begin{array}{l}
\frac{dx}{dt} = \rho_1 u_1(x, y)\sqrt{1-x} - \rho_2 u_2(x, y)\sqrt{x} \\
\frac{dy}{dt} = \rho_2 u_2(x, y)\sqrt{1-y} - \rho_1 u_1(x, y)\sqrt{y},
\end{array} \right. \tag{1.10}
\end{align}

where \(x, y\) - market share of the first and second firms sales, \(u_1, u_2\) - advertising spending of the first and second firms sales, \(\rho_1, \rho_2\) - advertising effectiveness of the first and second firm, \(m_1, m_2\) - marginal probit of the first and second firms sales, \(c_1, c_2\) - marginal cost of advertising of the first and second firms sales.

**Competitive stochastic advertising model (Prasad, 2003)**


\begin{align}
J_1(x(t)) = \max_{u_1(t) \geq 0} E \left[ \int_0^T e^{-rt}(m_1 x(t) - c_1 u_1(t)^2)dt \right]
\end{align}

\begin{align}
J_2(y(t)) = \max_{u_2(t) \geq 0} E \left[ \int_0^T e^{-rt}(m_2 y(t) - c_2 u_2(t)^2)dt \right] \tag{1.11}
\end{align}

\begin{align}
\left\{ \begin{array}{l}
\frac{dx}{dt} = \left[ \rho_1 u_1(x, y)\sqrt{1-x} - \rho_2 u_2(x, y)\sqrt{x} - \delta(x-y) \right] dt + \sigma(x, y)dw \\
x(0) = x_0 \\
\frac{dy}{dt} = \left[ \rho_2 u_2(x, y)\sqrt{1-y} - \rho_1 u_1(x, y)\sqrt{y} - \delta(y-x) \right] dt - \sigma(x, y)dw,
\end{array} \right. \tag{1.12}
\end{align}

where \(\delta_1, \delta_2\) - loss of market.

## 2 Extension of advertising model

**Theorem 3.** If \(\pi(0)\) is an initial profit and \(\pi(t) = mx(t) - cu(t)^2\) is an operating profit, then \(\gamma = -r\pi(T)e^{r(T-t)}\) is defined by the following problem,

\begin{align}
J(x(t), t) = \max_{u(t) \geq 0} E \left[ \int_0^T e^{-rt}(\gamma + mx(t) - cu(t)^2)dt \right] \tag{2.1}
\end{align}

**Proof.** According to the theorem 3 from (1.1)

\begin{align}
J(x, t) = E \left[ \int_{t_0}^T F(x(t), u(t), t)dt + \varphi(x(T)) \right] = E \left[ \int_t^T \pi(0)dt + \pi(T) \right],
\end{align}

where \(\pi(0) = \pi(t)e^{-rt}\)

\begin{align}
J(x(t), t) = E \left[ \int_t^T \pi(t)e^{-rt}dt - \frac{\gamma}{r}e^{-r(T-t)} \right] =
\end{align}
\[
E \left[ \int_t^T (mx(t) - cu(t)^2)e^{-rt}dt + \gamma \int_t^T e^{-rt}dt \right] = \\
= E \left[ \int_t^T e^{-rt}(\gamma + mx(t) - cu(t)^2)dt \right].
\]

This completes the proof if we take \( t = 0 \) and \( T = \infty \). \( \square \)

**Theorem 4.** Let the firm’s competitive response be in the form of the following Sorger model in the market only \((x, y)\).

\[
\begin{align*}
\frac{dx}{dt} &= \mu_1 u_1(x, y)\sqrt{1-x} - \mu_2 u_2(x, y)\sqrt{x} \\
\frac{dy}{dt} &= \mu_2 u_2(x, y)\sqrt{1-y} - \mu_1 u_1(x, y)\sqrt{y}
\end{align*}
\]

Then a competitive firm linear reaction can be expected as:

\[
\mu_1 = \nu_1 \mu_{11} + (1 - \nu_1) \mu_{12} \\
\mu_2 = \nu_2 \mu_{21} + (1 - \nu_2) \mu_{22}
\]

**Proof.** Substituting it into first equation of system we have

\[
\frac{dx}{dt} = (\nu_1 \mu_{11} + (1 - \nu_1) \mu_{12}) u_1(x, y)\sqrt{1-x} - (\nu_2 \mu_{21} + (1 - \nu_2) \mu_{22}) u_2(x, y)\sqrt{x} = \\
\nu_1 \mu_{11} u_1(x, y)\sqrt{1-x} - \nu_2 \mu_{21} u_2(x, y)\sqrt{x} + (1-\nu_1) \mu_{12} u_1(x, y)\sqrt{1-x} - (1-\nu_2) \mu_{22} u_2(x, y)\sqrt{x}.
\]

The system is written

\[
\begin{align*}
\frac{1}{2} \frac{dx}{dt} &= \nu_1 \mu_{11} u_1(x, y)\sqrt{1-x} - \nu_2 \mu_{21} u_2(x, y)\sqrt{x} \\
\frac{1}{2} \frac{dy}{dt} &= (1-\nu_1) \mu_{12} u_1(x, y)\sqrt{1-x} - (1-\nu_2) \mu_{22} u_2(x, y)\sqrt{x}.
\end{align*}
\]

Hence, we get

\[
\frac{dx}{dt} = -\frac{dy}{dt}.
\]

Substituting it into second equation;

\[
\begin{align*}
\frac{dx}{dt} &= 2\nu_1 \mu_{11} u_1(x, y)\sqrt{1-x} - 2\nu_2 \mu_{21} u_2(x, y)\sqrt{x} \\
\frac{dy}{dt} &= 2(1-\nu_2) \mu_{22} u_2(x, y)\sqrt{1-y} - 2(1-\nu_1) \mu_{12} u_1(x, y)\sqrt{y},
\end{align*}
\]

where

\[
\rho_{11} = 2\nu_1 \mu_{11}, \quad \rho_{21} = 2\nu_2 \mu_{21} u_2, \quad \rho_{22} = 2(1-\nu_2) \mu_{22}, \quad \rho_{12} = 2(1-\nu_1) \mu_{12}.
\]

This completes the proof. \( \square \)
Extension of advertising model is based on theorem 3 and 4. Consider a stochastic differential game model

\[
J_1(x(t)) = \max_{u_1(t) \geq 0} \mathbb{E} \left[ \int_0^\infty e^{-rt}(\gamma_1 + m_1 x(t) - c_1 u_1(t)^2)dt \right]
\]

\[
J_2(x(t)) = \max_{u_2(t) \geq 0} \mathbb{E} \left[ \int_0^\infty e^{-rt}(\gamma_2 + m_2 x(t) - c_2 u_2(t)^2)dt \right]
\]

\[
\begin{aligned}
&\begin{cases}
  dx = [\rho_{11} u_1(x, y) \sqrt{1 - x} - \rho_{21} u_2(x, y) \sqrt{1 - y} - \delta_1 (2x - 1)]dt + \theta_1 \sigma_1(x, y) dw_1 \\
  dy = [\rho_{22} u_2(x, y) \sqrt{1 - y} - \rho_{12} u_1(x, y) \sqrt{1 - x} - \delta_2 (2y - 1)]dt - \theta_2 \sigma_2(x, y) dw_2.
\end{cases}
\end{aligned}
\]

here \(\gamma_1, \gamma_2\)-coefficients of the first and second firms non-operating, \(\rho_{11}\)-effectiveness of the first firms advertising spending, \(\rho_{12}\)-reaction coefficient of the first firms market share to impact of the second firms market share, \(\rho_{22}\)-effectiveness of the second firms advertising spending, \(\rho_{21}\)-reaction coefficient of the second firms market share to impact of the first firms market share, \(\theta_1, \theta_2\)-coefficients of the first and second firms customer behavior, \(\sigma_1 dw_1, \sigma_2 dw_2\)- white noises of the first and second firms market share.

To find the closed-loop Nash Equilibrium strategies, we form the Hamilton-Jacobi-Bellman equation for each firm:

\[
\begin{aligned}
  r_1 J_1 &= \max_{u_1 \geq 0} \{\gamma_1 + m_1 x - c_1 u_1^2 + J_1'(\rho_{11} u_1 \sqrt{1 - x}) - \rho_{21} u_2 \sqrt{x} - \delta_1 (2x - 1) + \frac{\theta_1^2 \sigma_1^2 V''}{2}\} \\
  r_2 J_2 &= \max_{u_2 \geq 0} \{\gamma_2 + m_2 y - c_2 u_2^2 + J_2'(\rho_{22} u_2 \sqrt{1 - y}) - \rho_{12} u_1 \sqrt{y} - \delta_2 (2y - 1) + \frac{\theta_2^2 \sigma_2^2 V''}{2}\}
\end{aligned}
\]

\)

where \(J_1', J_2'\) and \(u_1^*, u_2^*\) denote the competitors advertising policies in (2.2), respectively. We obtain the optimal feedback advertising decisions

\[
\begin{aligned}
  u_1^*(x) &= \frac{J_1'(x) \rho_{11} \sqrt{1 - x}}{2c_1} = \frac{J_1'(x) \rho_{11} \sqrt{y}}{2c_1} \\
  u_2^*(x) &= \frac{J_2'(x) \rho_{22} \sqrt{1 - y}}{2c_2} = \frac{J_2'(x) \rho_{22} \sqrt{y}}{2c_2}.
\end{aligned}
\]

Following Sethi [30], we attempt to find forms for value functions \(J_1 = \alpha_1 + \beta_1 x\) and \(J_2 = \alpha_2 + \beta_2 y\). From equations (2.2) and (2.3), we determine the unknown coefficients \(x^0, x, y^0, y\).

\[
\begin{aligned}
  x^0 : \quad r_1 \alpha_1 &= \gamma_1 + \frac{\beta_1^2 \rho_{11}^2}{4c_1} + \beta_1 \delta_1 \\
  x : \quad r_1 \beta_1 &= m_1 - \frac{\beta_1^2 \rho_{11}^2}{4c_1} - \frac{\beta_1 \beta_2 \rho_{12} \rho_{22}}{2c_2} - 2 \beta_1 \delta_1 \\
  y^0 : \quad r_2 \alpha_2 &= \gamma_2 + \frac{\beta_2^2 \rho_{22}^2}{4c_2} + \beta_2 \delta_2 \\
  y : \quad r_2 \beta_2 &= m_2 - \frac{\beta_2^2 \rho_{22}^2}{4c_2} - \frac{\beta_1 \beta_2 \rho_{12} \rho_{21}}{2c_1} - 2 \beta_2 \delta_2
\end{aligned}
\]
Substituting \( W_1 = r_1 + 2\delta_1, \ W_2 = r_2 + 2\delta_2, \ R_2 = \rho_{22}/2c_2, \ R_1 = \rho_{11}/2c_1 \) into this system, we get
\[
\begin{align*}
W_1\beta_1 &= m_1 - \rho_{11}R_1\beta_1^2/2 - \rho_{21}R_2\beta_1\beta_2 \\
W_2\beta_2 &= m_2 - \rho_{22}R_2\beta_2^2/2 - \rho_{12}R_1\beta_1\beta_2
\end{align*}
\]

The system reduces to the following equation:
\[
\beta_1^4 + k_1\beta_1^3 + k_2\beta_1^2 - k_3\beta_1 - k_4 = 0 \tag{2.4}
\]
where \( r_1 > 0, \ r_2 > 0, \ \rho_{11} > 0, \ \rho_{21} > 0, \ \rho_{12} > 0, \ \rho_{22} > 0, \ m_1 > 0, \ m_2 > 0, \ c_1 > 0, \ c_2 > 0, \ \delta_1 > 0, \ \delta_2 > 0, \ W_1 = r_1 + 2\delta_1 > 0, \ W_2 = r_2 + 2\delta_2 > 0, \ R_2 = \rho_{22}/2c_2 > 0, \ R_1 = \rho_{11}/2c_1 > 0. \)

It can be easily checked that
\[
k_4 = \frac{\rho_{22}m_1^2}{R_1^2(\rho_{11}\rho_{12}\rho_{21} - \rho_{22}\rho_{11}^2/4)} > 0,
\]
\[
k_1 = \frac{(2\rho_{21}\rho_{12} - \rho_{22}\rho_{11})W_1 + \rho_{11}\rho_{21}W_2}{R_1^2(\rho_{11}\rho_{12}\rho_{21} - \rho_{22}\rho_{11}^2/4)} > 0,
\]
Of
\[
k_3 = \begin{cases} 
\frac{2m_1(\rho_{21}W_2 - \rho_{22}W_1)}{R_1^2(\rho_{11}\rho_{12}\rho_{21} - \rho_{22}\rho_{11}^2/4)} > 0, & \rho_{21}W_2 > \rho_{22}W_1 \\
\frac{2m_1(\rho_{22}W_2 - \rho_{22}W_1)}{R_1^2(\rho_{11}\rho_{12}\rho_{21} - \rho_{22}\rho_{11}^2/4)} < 0, & \rho_{21}W_2 < \rho_{22}W_1
\end{cases}
\]

This may be rewritten in its simplest form as (1.4)
\[
F(\beta_1) = \beta_1^4 + k_1\beta_1^3 + k_2\beta_1^2 \pm k_3\beta_1 - k_4 = 0
\]

For \( \beta_1 \rightarrow \pm\infty, \ F(\beta_1) \rightarrow \infty \) and when \( \beta_1 = 0, \ F(\beta_1) < 0. \)

Since \( F(\beta_1) \) is differentiable and continuous the above equation has a solution.

Thus optimal advertising is
\[
\begin{align*}
u_1^*(x) &= \frac{\bar{\beta}_1\rho_{11}\sqrt{T-x}}{2c_1} \\
u_2^*(x) &= \frac{\bar{\beta}_2\rho_{22}\sqrt{T-y}}{2c_2}
\end{align*}
\]

The state equation (2.1) becomes as:
\[
\begin{align*}
\frac{dx}{dt} &= [b_{11} - (b_{11} + b_{21})x]dt + \theta_1\sigma_1dw_1 \\
\frac{dy}{dt} &= [b_{22} - (b_{22} + b_{12})y]dt - \theta_2\sigma_2dw_2
\end{align*} \tag{2.5}
\]
where
\[
b_{11} = \frac{\rho_{11}^2}{2c_1} \bar{\beta}_1 + \delta_1, \ b_{21} = \frac{\rho_{21}\rho_{22}}{2c_2} \bar{\beta}_2 + \delta_1
\]
\[ b_{11} = \frac{\rho_{12}^2}{2\sigma_2^2} \beta_2 + \delta_2, \quad b_{12} = \frac{\rho_{12}\rho_{11}}{2\sigma_1} \beta_1 + \delta_2 \]

We search for solutions as \( z_1(t) = e^{(b_{11}+b_{21})t}x(t) \), \( z_2(t) = e^{(b_{22}+b_{12})t}y(t) \) and find solutions:

\[
\begin{align*}
\begin{cases}
  x(t) = x(0)e^{-(b_{11}+b_{21})t} + \frac{b_{11}}{b_{11}+b_{21}}((1 - e^{-(b_{11}+b_{21})t}) + \\
  \quad + \theta_1\sigma_1 \int_0^t e^{-(b_{11}+b_{21})(t-s)}dw_1(s))
  \\  y(t) = y(0)e^{-(b_{22}+b_{12})t} + \frac{b_{22}}{b_{22}+b_{12}}((1 - e^{-(b_{22}+b_{12})t}) + \\
  \quad - \theta_2\sigma_2 \int_0^t e^{-(b_{22}+b_{12})(t-s)}dw_2(s))
\end{cases}
\]

**Theorem 5.** Let \( f(t, x) \) be a function for which the partial derivatives \( f_t(t, x), f_x(t, x) \), and \( f_{xx}(t, x) \) are continuous. Then, for every \( T > 0 \), with the Ito process \( X(t) \) replacing the Brownian motion \( W(t) \), it holds:

\[
df(t, W(t)) = f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))dW(t)dW(t)
\]

A rigorous proof of this theorem can be found in Ito [15].

According to the theorem 5, let consider \( x^2(t), y^2(t) \). Then

\[
f(t, x(t)) = x^2(t) \Rightarrow dx^2(t) = 0 + 2x(t)dx(t) + \frac{1}{2}2dx(t)dx(t) = 2x(t)dx(t) + dx(t)dx(t)
\]

\[
f(t, y(t)) = y^2(t) \Rightarrow dy^2(t) = 0 + 2y(t)dy(t) + \frac{1}{2}2dy(t)dy(t) = 2y(t)dy(t) + dy(t)dy(t)
\]

We remind that \( dt^2 = 0, \; dt dw = 0, \) and \( dw dw = dt \)

\[
x^2(t) = x^2(0) + (2b_{11} + \theta_1^2\sigma_1^2) \int_0^t x(s)ds - (2(b_{11} + b_{21}) + \theta_1^2\sigma_1^2) \int_0^t x^2(s)ds +
\]

\[
\quad + 2\theta_1\sigma_1 \int_0^t x(s)\sqrt{x(s)(1 - x(s))}dw_1(s)
\]

\[
y^2(t) = y^2(0) + (2b_{22} + \theta_2^2\sigma_2^2) \int_0^t y(s)ds - (2(b_{22} + b_{12}) + \theta_2^2\sigma_2^2) \int_0^t y^2(s)ds +
\]

\[
\quad + 2\theta_2\sigma_2 \int_0^t y(s)\sqrt{y(s)(1 - y(s))}dw_2(s)
\]

Rewriting this as a stochastic integral, taking the expected value, and rewriting as a differential equation. From (2.6), we obtain a first order linear differential equation for the second moment \( E(x^2) \).

\[
\begin{align*}
\begin{cases}
  \frac{dE(x^2)}{dt} + (2(b_{11} + b_{21}) + \theta_1^2\sigma_1^2)E(x^2) = (2b_{11} + \theta_1^2\sigma_1^2)\frac{b_{11}}{b_{11} + b_{21}}
  \\  \frac{dE(y^2)}{dt} + (2(b_{22} + b_{12}) + \theta_2^2\sigma_2^2)E(y^2) = (2b_{22} + \theta_2^2\sigma_2^2)\frac{b_{22}}{b_{22} + b_{12}}
\end{cases}
\end{align*}
\]
The solution is
\[
\begin{aligned}
\lim_{t \to \infty} E(x^2) &= \frac{b_{11}(2b_{11}+\theta_{1}^2\sigma_{1}^2)}{(b_{11}+b_{21})(2(b_{11}+b_{21})+\theta_{1}^2\sigma_{1}^2)}, \\
\lim_{t \to \infty} E(y^2) &= \frac{b_{22}(2b_{22}+\theta_{2}^2\sigma_{2}^2)}{(b_{22}+b_{12})(2(b_{22}+b_{12})+\theta_{2}^2\sigma_{2}^2)},
\end{aligned}
\]
but the variance of the market shares in the long run are given by (2.6)
\[
\begin{aligned}
Var(x) &= \frac{b_{11}b_{21}(\theta_{1}^2\sigma_{1}^2)^2}{((b_{11}+b_{21})^2[(b_{11}+b_{21})+(\theta_{1}^2\sigma_{1}^2)^2])}, \\
Var(y) &= \frac{b_{22}b_{12}(\theta_{2}^2\sigma_{2}^2)^2}{((b_{22}+b_{12})^2[(b_{22}+b_{12})+(\theta_{2}^2\sigma_{2}^2)^2])}.
\end{aligned}
\]

**Theorem 6.** Let \( \pi_i(t) \) be a profit function with an exponential growth \( \pi_i(t) = \pi_i(0)e^{r_i t}, \) and \( i = 1, ..., n \). If value function is
\[
V_i(x(t)) = \max_{u_i(t) \geq 0} E \left[ \int_0^\infty e^{-r_i t}\pi_i(t)dt \right]
\]
Then, we can choose \( r_i \) so that \( r_i > \bar{r}_i \).

**Proof.** Clearly, \( V_i(x) > 0 \) for \( 0 < x < 1 \). We write down
\[
V_i(x(t)) = \max_{u_i(t) \geq 0} E \left[ \int_0^\infty e^{-r_i t}\pi_i(0)e^{r_i t}dt \right] = \max_{u_i(t) \geq 0} E \left[ \pi_i(0) \frac{1}{(r_i - \bar{r}_i)} \right] = \frac{\pi_i(0)}{r_i - \bar{r}_i}
\]
which implies that \( r_i > \bar{r}_i \). \( \square \)

### 3 Computational Results

We tested our proposed model on selected insurance companies of Mongolia such as Mongol, Mig, Bodi, Tenger, Practical, Ard, Gan-Zam, Munkh, Nomin, UB, Soyombo, Monnis, Mandal, Monre, Ger and Khan companies. Computational results are given in the following tables:
Tables 4.1, 4.2 and 4.3 parameters are calculated by other parameters.
### Table 4.4

<table>
<thead>
<tr>
<th>The parameters of the theory</th>
<th>Parameter results</th>
<th>Yes/no</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1 = \frac{\rho_{11}}{2c_1} &gt; 0$</td>
<td>$R_1 = \frac{2.71}{2 \cdot 56.532} = 0.0240$</td>
<td>Yes</td>
</tr>
<tr>
<td>$R_2 = \frac{\rho_{22}}{2c_2} &gt; 0$</td>
<td>$R_2 = \frac{2.0426}{2 \cdot 39.789} = 0.0257$</td>
<td>Yes</td>
</tr>
<tr>
<td>$W_1 = r_1 + 2\delta_1 &gt; 0$</td>
<td>$W_1 = 0.0226 + 2 \cdot 0.0342 = 0.0909$</td>
<td>Yes</td>
</tr>
<tr>
<td>$W_2 = r_2 + 2\delta_2 &gt; 0$</td>
<td>$W_2 = 0.0037 + 2 \cdot 0.0420 = 0.0876$</td>
<td>Yes</td>
</tr>
<tr>
<td>$\rho_{11}\rho_{12}\rho_{21} - \rho_{22}\rho_{11}^2/4 &gt; 0$</td>
<td>$2.71 \cdot 2.62 \cdot 1.80 - 2.04 \cdot \frac{2.71^2}{4} = 9.0356$</td>
<td>Yes</td>
</tr>
<tr>
<td>$k_1 = \frac{(2\rho_{23}\rho_{32} - \rho_{22}\rho_{31})W_1 + \rho_{11}\rho_{21}W_2}{R_1(\rho_{11}\rho_{12}\rho_{21} - \rho_{22}\rho_{11}^2/4)} &gt; 0$</td>
<td>$k_1 = 3.6120$</td>
<td>Yes</td>
</tr>
<tr>
<td>$k_4 = \frac{\rho_{22}m_1^2}{R_1^2(\rho_{11}\rho_{12}\rho_{21} - \rho_{22}\rho_{11}^2/4)} &gt; 0$</td>
<td>$k_4 = \frac{2.0426 \cdot 0.4503^2}{0.024^2 \cdot 9.0356} = 79.789$</td>
<td>Yes</td>
</tr>
<tr>
<td>$k_3 = \frac{2m_1(\rho_{21}W_2 - \rho_{22}W_1)}{R_1^2(\rho_{11}\rho_{12}\rho_{21} - \rho_{22}\rho_{11}^2/4)} &lt; 0$</td>
<td>$k_3 = -4.8322$</td>
<td>Yes</td>
</tr>
<tr>
<td>$k_2 = \frac{(2\rho_{21}W_2 - \rho_{22}W_1)W_1 + m_1R_1(\rho_{11}\rho_{22} - 2\rho_{21}\rho_{12}) + 2m_2\rho_{21}^2R_2}{R_1^2(\rho_{11}\rho_{12}\rho_{21} - \rho_{22}\rho_{11}^2/4)} = 1.4682 &gt; 0,$</td>
<td>$\text{Yes}$</td>
<td></td>
</tr>
</tbody>
</table>
One positive solution is $\bar{\beta}_1 = 2.016$.

\[
\bar{\beta}_2 = \frac{m_1 - \rho_{11} R_1 \beta_1^2/2 - W_1 \bar{\beta}_1}{\rho_{21} R_2 \bar{\beta}_1} = \frac{0.4503 - 2.71 \times 0.024 \times 2.0216^2/2 - 0.09 \times 2.0216}{1.8016 \times 0.0257 \times 2.0216} = 0.911
\]

and

\[
\alpha_1 = \frac{1}{r_1} \left[ \gamma_1 + \frac{\bar{\beta}_1^2 \rho_{11}^2}{4c_1} + \beta_1 \delta_1 \right] = \frac{1}{0.0226} \left[ \frac{2.2016^2 \times 2.71^2}{4 \times 56.532} + 2.0216 \times 0.0342 \right] = 10.29
\]

\[
\alpha_2 = \frac{1}{r_2} \left[ \gamma_2 + \frac{\bar{\beta}_2^2 \rho_{21}^2}{4c_2} + \beta_2 \delta_2 \right] = \frac{1}{0.0037} \left[ \frac{0.066 + 0.911^2 \times 1.8016^2}{4 \times 39.789} + 0.911 \times 0.042 \right] = 32.74.
\]

The valuation function was:

\[
\mathcal{J}_1 = 10.29 + 2.02x
\]

\[
\mathcal{J}_2 = 32.74 + 0.91y
\]

The market equilibrium levels is:

\[
b_{11} = \frac{\rho_{11}^2 \beta_1}{2c_1} + \delta_1 = \frac{2.71^2}{2 \times 56.532} \times 2.0216 + 0.0342 = 0.1772
\]

\[
b_{21} = \frac{\rho_{21} \rho_{22} \beta_2}{2c_2} + \delta_1 = \frac{1.8016 \times 2.0426}{2 \times 39.789} \times 0.911 + 0.0342 = 0.0763
\]

\[
b_{11} = \frac{\rho_{22}^2 \beta_2}{2c_2} + \delta_2 = \frac{2.0426^2}{2 \times 39.789} \times 0.911 + 0.042 = 0.0897
\]

\[
b_{12} = \frac{\rho_{12} \rho_{11} \beta_1}{2c_1} + \delta_2 = \frac{2.6188 \times 2.71}{2 \times 56.532} \times 2.0216 + 0.042 = 0.1801
\]

\[
\sigma_1 = \sqrt{\frac{2 \text{Var}(x)(b_{11} + b_{21})^3}{\theta_1^2(b_{11}b_{21} - \text{Var}(x)(b_{11} + b_{21}))}} = 0.112
\]

\[
\sigma_2 = \sqrt{\frac{2 \text{Var}(y)(b_{22} + b_{12})^3}{\theta_2^2(b_{22}b_{12} - \text{Var}(x)(b_{22} + b_{12}))}} = 0.056.
\]

Solutions to the differential system are:

\[
x(t) = x(0)e^{-(b_{11}+b_{21})t} + \frac{b_{11}}{b_{11} + b_{21}} ((1 - e^{-(b_{11}+b_{21})t}) + \theta_1 \sigma_1 \int_0^t e^{-(b_{11}+b_{21})(t-s)} dw_1(s) =
\]
\[
x(0)e^{-0.2534t} + 0.699((1 - e^{-0.2534t}) + 0.067 \int_0^t e^{-0.2534(t-s)}dw_1(s)
\]
\[
y(t) = y(0)e^{-(b_{22}+b_{12})t} + \frac{b_{22}}{b_{22}+b_{12}}((1 - e^{-(b_{22}+b_{12})t}) + -\theta_2\sigma_2 \int_0^t e^{-(b_{22}+b_{12})(t-s)}dw_2(s) =
\]
\[
y(t) = y(0)e^{-0.2699t} + 0.3325((1 - e^{-0.2699t}) + 0.067 \int_0^t e^{-(b_{22}+b_{12})(t-s)}dw_2(s)
\]

A large insurance company is 69.9 percent of the market equilibrium, while a small insurance company has 33.3 percent of the market equilibrium. Optimal advertising spending of a large insurance company is

\[
u_1^*(x) = \frac{\beta_1\rho_{11}\sqrt{1-x}}{2c_1} = \frac{2.0216 \times 2.71\sqrt{1-0.699}}{2 \times 56.532} = 0.027
\]

Optimal advertising spending of a small insurance company is

\[
u_2^*(x) = \frac{\beta_2\rho_{22}\sqrt{1-y}}{2c_2} = \frac{0.911 \times 2.0426\sqrt{1-0.3325}}{2 \times 39.789} = 0.019
\]

obtained respectively.

References


GENERALIZATIONS OF SHERMAN’S INEQUALITY BY MONTGROMEY IDENTITY

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Abstract. In this paper, we give some general identity for the difference of Sherman’s inequality by using Montgromey identity. We establish generalized Sherman’s inequality for n-convex function. We use Čebyšev functionals and give bounds for the identities related to the generalization of Sherman’s inequality. We also give Grüss type inequalities and Ostrowski-type inequalities for these functionals. We present mean value theorems and n-exponential convexity which leads to exponential convexity and then log-convexity for these functionals. We give some families of functions which enable us to construct a large families of functions that are exponentially convex and also give Stolarsky type means.

1. Introduction

For fixed \(m \geq 2\), let \(x = (x_1, \ldots, x_m)\) and \(y = (y_1, \ldots, y_m)\) denote two \(m\)-tuples. Let

\[
\begin{align*}
&x[1] \geq x[2] \geq \ldots \geq x[m], \quad y[1] \geq y[2] \geq \ldots \geq y[m], \\
&x(1) \leq x(2) \leq \ldots \leq x(m), \quad y(1) \leq y(2) \leq \ldots \leq y(m)
\end{align*}
\]

be their ordered components. We say that \(x\) majorizes \(y\) or \(y\) is majorized by \(x\) and write

\[y \prec x\]

if

\[
\begin{align*}
\sum_{i=1}^{k} y[i] &\leq \sum_{i=1}^{k} x[i], \quad k = 1, \ldots, m - 1, \\
\sum_{i=1}^{m} y[i] &\leq \sum_{i=1}^{m} x[i].
\end{align*}
\]

Note that (1.1) is equivalent to

\[
\begin{align*}
\sum_{i=m-k+1}^{m} y(i) &\leq \sum_{i=m-k+1}^{m} x(i), \quad k = 1, \ldots, m - 1.
\end{align*}
\]
The following notion of Schur-convexity generalizes the definition of convex function via the notion of majorization.

A function $F : S \subseteq \mathbb{R}^m \to \mathbb{R}$ is called Schur-convex on $S$ if

$$F(y) \leq F(x)$$

for every $x, y \in S$ such that $y \prec x$.

A relation between one-dimensional convex function and $m$-dimensional Schur-convex function is included in the following Majorization theorem proved by G. H. Hardy, J. E. Littlewood, G. Pólya (see [6], [12, p. 333]).

**Theorem 1** (Majorization theorem). Let $I \subset \mathbb{R}$ be an interval and $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in I^m$. Let $f : I \to \mathbb{R}$ be continuous function. Then a function $F : I^m \to \mathbb{R}$, defined by

$$F(x) = \sum_{i=1}^{m} f(x_i),$$

is Schur-convex on $I^m$ iff $f$ is convex on $I$.

The following theorem gives weighted generalization of Majorization theorem (see [5], [12, p. 323]).

**Theorem 2** (Fuchs’s theorem). Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in I^m$ be two decreasing $m$-tuples and $p = (p_1, \ldots, p_m)$ be a real $m$-tuple such that

$$\sum_{i=1}^{k} p_i y_i \leq \sum_{i=1}^{k} p_i x_i, \quad k = 1, \ldots, m - 1,$$

$$\sum_{i=1}^{m} p_i y_i = \sum_{i=1}^{m} p_i x_i.$$

Then for every continuous convex function $f : I \to \mathbb{R}$, we have

$$\sum_{i=1}^{m} p_i f(y_i) \leq \sum_{i=1}^{m} p_i f(x_i).$$

The Jensen inequality in the form

$$f \left( \frac{1}{P_m} \sum_{i=1}^{m} p_i x_i \right) \leq \frac{1}{P_m} \sum_{i=1}^{m} p_i f(x_i)$$

(1.3)

for convex function $f$, where $p = (p_1, \ldots, p_m)$ is a nonnegative $m$-tuple such that $P_m = \sum_{i=1}^{m} p_i > 0$, can be obtained as a special case of the previous result putting $y_1 = y_2 = \ldots = y_m = \frac{1}{P_m} \sum_{i=1}^{m} p_i x_i$.

A natural problem of interest is extension of notation from $m$-tuples (vectors) to $m \times l$ matrices $A = (a_{ij}) \in M_{ml}(\mathbb{R})$. In order that, we introduce the notion of row stochastic and double stochastic matrices.

A matrix $A = (a_{ij}) \in M_{ml}(\mathbb{R})$ is called row stochastic if all of its entries are greater or equal to zero, i.e. $a_{ij} \geq 0$ for $i = 1, \ldots, m, \, j = 1, \ldots, l$ and the sum of
the entries in each row is equal to 1, i.e. \( \sum_{j=1}^{l} a_{ij} = 1 \) for \( i = 1, \ldots, m \). If in addition
the transpose \( A^T = (a_{ji}) \) of \( A = (a_{ij}) \) is row stochastic, then \( A \) is called doubly stochastic. In other words, \( A = (a_{ij}) \in M_l(\mathbb{R}) \) is called double stochastic if all of
its entries are greater or equal to zero (nonnegative), i.e. \( a_{ij} \geq 0 \) for \( i = 1, \ldots, m, j = 1, \ldots, l \) and the sum of the entries in each column and each row is equal to 1,
i.e. \( \sum_{i=1}^{l} a_{ij} = 1 \) for \( j = 1, \ldots, l \) and \( \sum_{j=1}^{l} a_{ij} = 1 \) for \( i = 1, \ldots, l \).

It is well known that for \( x, y \in \mathbb{R}^l \) is valid
\[
y < x \quad \text{if and only if} \quad y = xA
\]
for some double stochastic matrix \( A \in M_l(\mathbb{R}) \).

The next generalization is obtained by S. Sherman (see [10], [14]).

**Theorem 3** (Sherman’s theorem). Let \( x = (x_1, \ldots, x_l) \in [\alpha, \beta]^l \), \( y = (y_1, \ldots, y_m) \in [\alpha, \beta]^m \), \( u = (u_1, \ldots, u_l) \in [0, \infty]^l \), \( v = (v_1, \ldots, v_m) \in [0, \infty]^m \) and
\[
y = xA^T \quad \text{and} \quad u = vA \tag{1.4}
\]
for some row stochastic matrix \( A = (a_{ij}) \in M_{ml}(\mathbb{R}) \). Then for every convex function \( f : [\alpha, \beta] \to \mathbb{R} \) we have
\[
\sum_{i=1}^{m} v_i f(y_i) \leq \sum_{j=1}^{l} u_j f(x_j). \tag{1.5}
\]

**Remark 1.** In a special case, from Sherman’s theorem we get Fuchs’s theorem. When \( m = l \), and all weights \( b_i \) and \( a_j \) are equal and nonnegative, the condition \( a = bA \) assures the stochasticity on columns, so in that case we deal with doubly stochastic matrices.

In order to obtain our main results in the present paper, we use the generalized Montgomery identity via Taylor’s formula given in paper [3].

**Theorem 4.** Let \( n \in \mathbb{N} \), \( \phi : I \to \mathbb{R} \) be such that \( \phi^{(n-1)} \) is absolutely continuous, \( I \subset \mathbb{R} \) an open interval, \( \alpha, \beta \in I \) and \( \alpha \leq \beta \). Then the following identity holds
\[
\phi(x) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi(t) \, dt + \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(\alpha)}{k! (k+2)} (x - \alpha)^{k+2} \frac{1}{\beta - \alpha} - \sum_{k=0}^{n-2} \frac{\phi^{(k+1)}(\beta)}{k! (k+2)} (x - \beta)^{k+2} \frac{1}{\beta - \alpha} + \frac{1}{(n-1)!} \int_{\alpha}^{\beta} T_n(x, s) \phi^{(n)}(s) \, ds \tag{1.6}
\]
where
\[
T_n(x, s) = \begin{cases} 
-\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{s-\alpha}{\beta-\alpha} (x-s)^{n-1}, & \alpha \leq s \leq x, \\
-\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\beta}{\beta-\alpha} (x-s)^{n-1}, & x < s \leq \beta.
\end{cases} \tag{1.7}
\]

To complete the Introduction, we state definition of divided differences and \( n \)-convexity (see for example [12]).
Definition 1. The divided difference of order \( n \), \( n \in \mathbb{N} \), of the function \( \phi : [\alpha, \beta] \to \mathbb{R} \) at mutually different points \( x_0, x_1, ..., x_n \in [\alpha, \beta] \) is defined recursively by

\[
[x; \phi] = \phi(x), \quad i = 0, ..., n
\]

\[
[x_0, ..., x_n; \phi] = \frac{x_1, ..., x_n; \phi - [x_0, ..., x_{n-1}; \phi]}{x_n - x_0}.
\]

The value \([x_0, ..., x_n; \phi]\) is independent of the order of the points \( x_0, ..., x_n \).

This definition may be extended to include the case in which some or all the points coincide. Assuming that \( f^{(j-1)}(x) \) exists, we define

\[
[x, ..., x; \phi] = \frac{\phi^{(j-1)}(x)}{(j-1)!}, \quad (1.8)
\]

Definition 2. A function \( \phi : [\alpha, \beta] \to \mathbb{R} \) is \( n \)-convex, \( n \geq 0 \), if for all choices of \((n + 1)\) distinct points \( x_i \in [\alpha, \beta], i = 0, ..., n \), the inequality

\[
[x_0, x_1, ..., x_n; \phi] \geq 0
\]

holds.

From Definition 2, it follows that 2-convex functions are just convex functions. Furthermore, 1-convex functions are increasing functions and 0-convex functions are nonnegative functions.

2. Main results

Theorem 5. Let \( n \in \mathbb{N} \), \( \phi : I \to \mathbb{R} \) be such that \( \phi^{(n-1)} \) is absolutely continuous, \( I \subset \mathbb{R} \) an open interval, \( \alpha, \beta \in I \), \( \alpha < \beta \). Suppose that \( x = (x_1, ..., x_l) \in [\alpha, \beta]^l \), \( y = (y_1, ..., y_m) \in [\alpha, \beta]^m \), \( u = (u_1, ..., u_l) \in \mathbb{R}^l \), \( v = (v_1, ..., v_m) \in \mathbb{R}^m \) be such that (1.4) holds for some matrix \( A = (a_{ij}) \in \mathcal{M}_{lm}(\mathbb{R}) \) satisfying the condition

\[
\sum_{j=1}^m a_{ij} = 1, \quad i = 1, 2, ..., l.
\]

Then

\[
\sum_{p=1}^l u_p \phi(x_p) - \sum_{q=1}^m v_q \phi(y_q)
\]

\[
= \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k!(k+2)} \left[ \phi^{(k+1)}(\alpha) \left( \sum_{p=1}^l u_p (x_p - \alpha)^{k+2} - \sum_{q=1}^m v_q (y_q - \alpha)^{k+2} \right) \right. 
\]

\[
- \phi^{(k+1)}(\beta) \left( \sum_{p=1}^l u_p (x_p - \beta)^{k+2} - \sum_{q=1}^m v_q (y_q - \beta)^{k+2} \right) \right] 
\]

\[
+ \frac{1}{(n-1)!} \int_{\alpha}^{\beta} \left( \sum_{p=1}^l u_p T_n(x_p, s) - \sum_{q=1}^m v_q T_n(y_q, s) \right) \phi^{(n)}(s) ds.
\]

(2.1)

Proof. Use (1.6) in \( \sum_{p=1}^l u_p \phi(x_p) - \sum_{q=1}^m v_q \phi(y_q) \) we obtain (2.1). \( \square \)

Now we state the main generalization of the Sherman inequality by using the above obtained identity.
Theorem 6. Let \( n \in \mathbb{N} \), \( \phi : I \to \mathbb{R} \) be such that \( \phi^{(n-1)} \) is absolutely continuous, \( I \subset \mathbb{R} \) an open interval, \( \alpha, \beta \in I \), \( \alpha < \beta \). Suppose that \( x = (x_1, ..., x_l) \in [\alpha, \beta]^l \), \( y = (y_1, ..., y_m) \in [\alpha, \beta]^m \), \( u = (u_1, ..., u_l) \in \mathbb{R}^l \), \( v = (v_1, ..., v_m) \in \mathbb{R}^m \) be such that (1.4) holds for some matrix \( A = (a_{ij}) \in \mathcal{M}_{lm}(\mathbb{R}) \) satisfying the condition \( \sum_{j=1}^{m} a_{ij} = 1 \), \( i = 1, 2, ..., l \). For \( T_n \) as defined in (1.7) assume that

\[
\sum_{q=1}^{m} v_q T_n(y_p, s) \leq \sum_{p=1}^{l} u_p T_n(x_p, s). \tag{2.2}
\]

Then for every \( n \)-convex function \( \phi : I \to \mathbb{R} \) the following inequality holds

\[
\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q) \\
\geq \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k!(k+2)} \left[ \phi^{(k+1)}(\alpha) \left( \sum_{p=1}^{l} u_p(x_p - \alpha)^{k+2} - \sum_{q=1}^{m} v_q(y_q - \alpha)^{k+2} \right) \right] \\
- \phi^{(k+1)}(\beta) \left( \sum_{p=1}^{l} u_p(x_p - \beta)^{k+2} - \sum_{q=1}^{m} v_q(y_q - \beta)^{k+2} \right). \tag{2.3}
\]

Proof. Since the function \( \phi \) is \( n \)-convex so we have \( \phi^{(n)} \geq 0 \) using this fact and (2.3) we easily arrive at our required result. \( \square \)

Theorem 7. Let \( n \in \mathbb{N} \), \( \phi : I \to \mathbb{R} \) be such that \( \phi^{(n-1)} \) is absolutely continuous, \( I \subset \mathbb{R} \) an open interval, \( \alpha, \beta \in I \), \( \alpha < \beta \). Let \( x = (x_1, x_2, ..., x_l) \in [\alpha, \beta]^l \), \( y = (y_1, y_2, ..., y_m) \in [\alpha, \beta]^m \), \( u = (u_1, u_2, ..., u_l) \in [0, \infty]^l \) and \( v = (v_1, v_2, ..., v_m) \in [0, \infty]^m \) be such that (1.4) holds for some row stochastic matrix \( A = (a_{ij}) \in \mathcal{M}_{lm}(\mathbb{R}) \). If \( \phi \) is \( 2n \)-convex function. Then

\[
\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q) \\
\geq \frac{1}{\beta - \alpha} \sum_{k=0}^{2n-2} \frac{1}{k!(k+2)} \left[ \phi^{(k+1)}(\alpha) \left( \sum_{p=1}^{l} u_p(x_p - \alpha)^{k+2} - \sum_{q=1}^{m} v_q(y_q - \alpha)^{k+2} \right) \right] \\
- \phi^{(k+1)}(\beta) \left( \sum_{p=1}^{l} u_p(x_p - \beta)^{k+2} - \sum_{q=1}^{m} v_q(y_q - \beta)^{k+2} \right). \tag{2.4}
\]

Moreover if \( \phi^{(k)}(\alpha) \geq 0 \) and \( (-1)^k \phi^{(k)}(\beta) \geq 0 \) for \( k = 1, ..., 2n - 1 \), then

\[
\sum_{p=1}^{l} u_p \phi(x_p) \geq \sum_{q=1}^{m} v_q \phi(y_q).
\]

Proof. Since

\[
T_n(x, s) = \begin{cases} 
-\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\alpha}{\beta-\alpha} (x-s)^{n-1}, & \alpha \leq s \leq x, \\
-\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\beta}{\beta-\alpha} (x-s)^{n-1}, & x < s \leq \beta.
\end{cases} \tag{2.5}
\]
and
\[
d^2T_n(x,s) = \begin{cases} \frac{n-1}{\beta-\alpha}[(x-s)^{n-2} + (n-2)(x-\alpha)(x-s)^{n-3}], & \alpha \leq s \leq x \\ \frac{n-1}{\beta-\alpha}[(x-s)^{n-2} + (n-2)(x-\beta)(x-s)^{n-3}], & \alpha \leq x \leq s \leq \beta. \end{cases} \tag{2.6}
\]

Therefore \(T_n(.,s)\) is continues for every \(n \geq 2\) and convex if \(n\) is even. Hence by Sherman’s Theorem the inequality (2.2) holds for even \(n\) and by using Theorem 6 the inequality (2.4) hold if \(n\) is replaced by \(2n\).

To prove the second part let \(f(x) = (x-\alpha)^{k+2}, x \in [\alpha, \beta]\) implies that \(f(x)\) is continues convex function. Again using the Sherman’s Theorem for the function \(f\) we get
\[
\sum_{p=1}^{l} u_p(x_p - \alpha)^{k+2} \geq \sum_{q=1}^{m} v_q(y_q - \alpha)^{k+2}. \tag{2.7}
\]

Similarly the function \(g(x) = (x-\beta)^{k+2}, x \in [\alpha, \beta]\), is convex if \(k\) is even and concave if \(k\) is odd, so we have
\[
\sum_{p=1}^{l} u_p(x_p - \beta)^{k+2} \geq \sum_{q=1}^{m} v_q(y_q - \beta)^{k+2}, \quad \text{if } k \text{ is even} \tag{2.8}
\]
\[
\sum_{p=1}^{l} u_p(x_p - \beta)^{k+2} \leq \sum_{q=1}^{m} v_q(y_q - \beta)^{k+2}, \quad \text{if } k \text{ is odd}. \tag{2.9}
\]

Now using (2.7), (2.8), (2.9) and the the assumptions \(\phi^{(k)}(\alpha) \geq 0\) and \((-1)^k \phi^{(k)}(\beta) \geq 0, k = 1, 2, \ldots, 2n - 1\), we have
\[
\frac{1}{\beta - \alpha} \sum_{k=0}^{2n-2} \left\{ \sum_{p=1}^{l} u_p(x_p - \alpha)^{k+2} - \sum_{q=1}^{m} v_q(y_q - \alpha)^{k+2} \right\} \frac{\phi^{(k+1)}(\alpha)}{k!(k+2)} \\
- \frac{1}{\beta - \alpha} \sum_{k=0}^{2n-2} \left\{ \sum_{p=1}^{l} u_p(x_p - \beta)^{k+2} - \sum_{q=1}^{m} v_q(y_q - \beta)^{k+2} \right\} \frac{\phi^{(k+1)}(\beta)}{k!(k+2)} \geq 0.
\]

Hence the result follows. \(\square\)

3. Bounds for identities related to generalization of Sherman’s inequality

For two Lebesgue integrable functions \(f, g : [\alpha, \beta] \to \mathbb{R}\), we consider the Čebyšev functional:
\[
\Lambda(f,g) := \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(t)g(t)dt - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(t)dt \cdot \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} g(t)dt.
\]

We use the following two theorems, proved in [4], to obtain generalizations of the results from the previous section.
Using Čebyšev functional we obtain bounds for identities related to generalization of Sherman’s inequality by Montgomery identity.

**Theorem 8.** Let \( f : [\alpha, \beta] \to \mathbb{R} \) be Lebesgue integrable and \( g : [\alpha, \beta] \to \mathbb{R} \) be absolutely continuous with \((\cdot - \alpha)(\beta - \cdot)(g')^2 \in L[\alpha, \beta]\). Then

\[
|\Lambda(f, g)| \leq \frac{1}{\sqrt{2}} |\Lambda(f, f)| \frac{1}{\sqrt{\beta - \alpha}} \left( \int_{\alpha}^{\beta} (x - \alpha)(\beta - x)(g'(x))^2 \, dx \right)^{\frac{1}{2}}. \tag{3.1}
\]

The constant \( \frac{1}{\sqrt{2}} \) in (3.1) is the best possible.

**Theorem 9.** Let \( g : [\alpha, \beta] \to \mathbb{R} \) be monotonic nondecreasing and \( f : [\alpha, \beta] \to \mathbb{R} \) be absolutely continuous with \( f' \in L_{\infty}[\alpha, \beta] \). Then

\[
|\Lambda(f, g)| \leq \frac{1}{2(\beta - \alpha)} \| f' \|_{\infty} \int_{\alpha}^{\beta} (x - \alpha)(\beta - x) \, dg(x). \tag{3.2}
\]

The constant \( \frac{1}{2} \) in (3.2) is the best possible.

For the sake of simplicity and to avoid many notations, we define two functions as follows.

Let \( x = (x_1, \ldots, x_l) \in [\alpha, \beta]^l \), \( y = (y_1, \ldots, y_k) \in [\alpha, \beta]^k \), \( u = (u_1, \ldots, u_l) \in \mathbb{R}^l \), \( v = (v_1, \ldots, v_k) \in \mathbb{R}^k \) and the function \( T_n \) be as defined in (1.7) respectively. Consider the function \( \mathcal{R}_H : [\alpha, \beta] \to \mathbb{R} \) defined by

\[
\mathcal{R}_H(s) = \sum_{p=1}^{l} u_p T_n(x_p, s) - \sum_{q=1}^{m} v_q T_n(y_p, s). \tag{3.3}
\]

Using Čebyšev functional we obtain bounds for identities related to generalization of Sherman’s inequality.

**Theorem 10.** Let \( \phi : [\alpha, \beta] \to \mathbb{R} \) be such that \( \phi \in C^m([\alpha, \beta]) \) with \((\cdot - \alpha)(\beta - \cdot)(\phi^{(n+1)})^2 \in L[\alpha, \beta]\) and \( x = (x_1, \ldots, x_l) \in [\alpha, \beta]^l \), \( y = (y_1, \ldots, y_k) \in [\alpha, \beta]^k \), \( u = (u_1, \ldots, u_l) \in \mathbb{R}^l \), \( v = (v_1, \ldots, v_k) \in \mathbb{R}^k \) and let the function \( \mathcal{R}_H \) be defined as in (3.3). Then

\[
\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{k} v_q \phi(y_q) = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k!(k+2)} \left[ \phi^{(k+1)}(\alpha) \left( \sum_{p=1}^{l} u_p (x_p - \alpha)^{k+2} - \sum_{q=1}^{m} v_q (y_q - \alpha)^{k+2} \right) \right.
\]

\[
- \phi^{(k+1)}(\beta) \left( \sum_{p=1}^{l} u_p (x_p - \beta)^{k+2} - \sum_{q=1}^{m} v_q (y_q - \beta)^{k+2} \right) \left. \right] + \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{\beta - \alpha} \int_{\alpha}^{\beta} \mathcal{R}_H(s) \, ds + \kappa_H(\phi; \alpha, \beta), \tag{3.4}
\]

where the remainder \( \kappa_H(\phi; \alpha, \beta) \) satisfies the estimation

\[
|\kappa_H(\phi; \alpha, \beta)| \leq \frac{\sqrt{\beta - \alpha}}{\sqrt{2}} \left[ \Lambda(\mathcal{R}_H, \mathcal{R}_H) \right] \left( \int_{\alpha}^{\beta} (s - \alpha)(\beta - s)(\phi^{(n+1)}(s))^2 \, ds \right)^{\frac{1}{2}}. \tag{3.5}
\]

**Proof.** The proof is similar to the proof of Theorem 15 in [1]. \( \square \)
Using Theorem 9 we obtain the following Grüss type inequality.

**Theorem 11.** Let $\phi \in C^n([\alpha, \beta])$ such that $\phi^{(n)}$ is monotonic non decreasing on $[\alpha, \beta]$ and let $R_H$ be defined by (3.3). Then the representation (3.4) holds and the remainder $\kappa_H(\phi; \alpha, \beta)$ satisfies the bound

$$|\kappa_H(\phi; \alpha, \beta)| \leq \|R_H\|_\infty \left\{ \frac{\phi^{(n-1)}(\beta) + \phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta) - \phi^{(n-2)}(\alpha)}{\beta - \alpha} \right\}.$$  

(3.6)

**Proof.** The proof is similar to the proof of Theorem 17 in [1]. □

Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a function then the $p$-norm of $\phi$ is defined by

$$\|\phi\|_p = \left\{ \left( \int_\alpha^\beta |\phi(t)|^p dt \right)^{\frac{1}{p}} \right\}, \text{ for } 1 \leq p < \infty,$$

if $|\phi|^p$ is $R$-integrable function,

$$\text{essential supremum of } \phi, \text{ for } p = \infty,$$

if $\phi$ is essential bounded.

We present the Ostrowsky-type inequalities related to generalization of Sherman’s inequality.

**Theorem 12.** Suppose that all assumptions of Theorem 5 hold. Assume $(p, q)$ is a pair of conjugate exponents, that is $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Let $|\phi^{(n)}|^p : [\alpha, \beta] \to \mathbb{R}$ be an $R$-integrable function. Then we have:

$$\left| \sum_{p=1}^l u_p \phi(x_p) - \sum_{q=1}^k v_q \phi(y_q) \right|$$

$$- \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k!(k+2)} \left[ \phi^{(k+1)}(\alpha) \left( \sum_{p=1}^l u_p (x_p - \alpha)^{k+2} - \sum_{q=1}^m v_q (y_q - \alpha)^{k+2} \right) \right]$$

$$- \frac{1}{\beta} \phi^{(k+1)}(\beta) \left( \sum_{p=1}^l u_p (x_p - \beta)^{k+2} - \sum_{q=1}^m v_q (y_q - \beta)^{k+2} \right) \right| \leq \|\phi^{(n)}\|_p \|R_H\|_q,$$

(3.7)

where $R_H$ is defined in (3.3).

The constant on the right-hand side of (3.7) is sharp for $1 < p \leq \infty$ and the best possible for $p = 1$.

**Proof.** The proof is similar to the proof of Theorem 19 in [1]. □

4. MEAN VALUE THEOREMS AND EXPONENTIAL CONVEXITY

In this section, we present the Lagrange and the Cauchy type of mean-value theorems using results from the previous section. Also, we use so called *Exponential convexity method* established in [7] in order to interpret our results in the form of exponentially convex functions or in the special case logarithmically convex functions. For some related results see also [9], [11].
Motivated by the inequality (2.3), under the assumptions of Theorems 6, we define the following linear functional:

\[ F^H(\phi) = \sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q) \]

\[ - \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k!(k+2)} \left( \phi^{(k+1)}(\alpha) \left( \sum_{p=1}^{l} u_p (x_p - \alpha)^{k+2} - \sum_{q=1}^{m} v_q (y_q - \alpha)^{k+2} \right) \right. \]

\[ - \phi^{(k+1)}(\beta) \left( \sum_{p=1}^{l} u_p (x_p - \beta)^{k+2} - \sum_{q=1}^{m} v_q (y_q - \beta)^{k+2} \right) \left. \right) \] \hspace{1cm} (4.1)

**Remark 2.** Under the assumptions of Theorems 6, it holds \( F^H(\phi) \geq 0 \), for all \( n \)-convex functions \( \phi \).

**Theorem 13.** Let \( \phi, \psi \in C^n([\alpha, \beta]) \). If the inequality in (2.3) holds, then there exists \( \xi \in [\alpha, \beta] \) such that

\[ F^H(\phi) = \phi^{(n)}(\xi) F^H(\psi), \]

where \( \varphi(x) = \frac{x^n}{n!} \) and \( F^H \) is defined by (4.1).

*Proof.* Similar to the proof of Theorem 4.1 in [8]. \( \square \)

**Theorem 14.** Let \( \phi, \psi \in C^n([\alpha, \beta]) \). If the inequality in (2.3) holds, then there exists \( \xi \in [\alpha, \beta] \) such that

\[ \frac{F^H(\phi)}{F^H(\psi)} = \frac{\phi^{(n)}(\xi)}{\psi^{(n)}(\xi)}, \]

provided that the denominators are non-zero and \( F^H \) is defined by (4.1).

*Proof.* Similar to the proof of Corollary 4.2 in [8]. \( \square \)

**Remark 3.** If \( \frac{\phi^{(n)}}{\psi^{(n)}} \) is an invertible function, then from (4.2) it follows

\[ \xi = \left( \frac{\phi^{(n)}}{\psi^{(n)}} \right)^{-1} \left( \frac{F^H(\phi)}{F^H(\psi)} \right). \]

Trough the rest of paper, \( I \) denotes an open interval in \( \mathbb{R} \).

Exponentially convex functions have many nice properties. Here we point some of them that we use in sequel.

**Definition 3.** For fixed \( n \in \mathbb{N} \), a function \( f : I \to \mathbb{R} \) is \( n \)-exponentially convex in the Jensen sense on \( I \) if

\[ \sum_{i,j=1}^{n} p_i p_j f \left( \frac{x_i + x_j}{2} \right) \geq 0 \]

holds for all choices \( p_i \in \mathbb{R} \) and \( x_i \in I, \ i = 1, \ldots, n \).

A function \( f : I \to \mathbb{R} \) is \( n \)-exponentially convex on \( I \) if it is \( n \)-exponentially convex in the Jensen sense and continuous on \( I \).

The notation of \( n \)-exponential convexity is introduced in [11].
Remark 4. From Definition 3 it follows that 1-exponentially convex functions in the Jensen sense are exactly nonnegative functions. Also, \( n \)-exponentially convex functions in the Jensen sense are \( k \)-exponentially convex in the Jensen sense for every \( k \in \mathbb{N}, k \leq n \).

Using basic calculus and definition of positive semidefinite matrices we have the following proposition.

**Proposition 1.** If \( f : I \to \mathbb{R} \) is an \( n \)-exponentially convex function in the Jensen sense then the matrix

\[
\begin{bmatrix}
  f \left( \frac{x_i + x_j}{2} \right)
\end{bmatrix}_{i,j=1}^k
\]

is positive semi-definite. Particularly,

\[
\det \begin{bmatrix}
  f \left( \frac{x_i + x_j}{2} \right)
\end{bmatrix}_{i,j=1}^k \geq 0
\]

holds for all \( k \in \mathbb{N}, k \leq n \) and \( x_i \in I, i = 1, \ldots, k \).

**Definition 4.** A function \( f : I \to \mathbb{R} \) is exponentially convex in the Jensen sense on \( I \) if it is \( n \)-exponentially convex in the Jensen sense for all \( n \in \mathbb{N} \).

One of the most important properties of exponentially convex functions is their integral representation (see [2, p. 211]).

**Theorem 15.** The function \( f : I \to \mathbb{R} \) is exponentially convex on \( I \) if and only if

\[
f(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(s), \tag{4.3}
\]

for some non-decreasing function \( \sigma : \mathbb{R} \to \mathbb{R} \).

We give some examples of exponentially convex functions that we use in sequel. For more details see [7].

**Example 1.** The basic example of exponentially convex function is the function \( f : I \to \mathbb{R} \) defined by \( f(x) = ce^{kx} \) for every \( c \geq 0 \) and \( k \in \mathbb{R} \).

The next examples can be deduced using integral representation (4.3) and some results of the Laplace transform.

**Example 2.**

(ii) The function \( f : (0, \infty) \to \mathbb{R} \) defined by \( f(x) = x^{-k} \) is exponentially convex on \((0, \infty)\) for every \( k > 0 \) since \( x^{-k} = \int_0^{\infty} e^{-xt} \frac{t^{-k-1}}{k} dt \) (see [13, p. 210]).

(iii) The function \( f : (0, \infty) \to (0, \infty) \) defined by \( f(x) = e^{-k\sqrt{x}} \) is exponentially convex on \((0, \infty)\) for every \( k > 0 \) since \( e^{-k\sqrt{x}} \int_0^{\infty} e^{-xt} e^{-k^2/4t} \frac{k}{2\sqrt{\pi}t^{3/2}} dt \) (see [13, p. 214]).

**Definition 5.** A function \( f : I \to (0, \infty) \) is said to be logarithmically convex in the Jensen sense if

\[
f \left( \frac{x + y}{2} \right) \leq \sqrt{f(x)f(y)}
\]

holds for all \( x, y \in I \).
**Definition 6.** A function $f : I \to (0, \infty)$ is said to be logarithmically convex or log-convex if
\[
f ((1 - \lambda)s + \lambda t) \leq f(s)^{1-\lambda}f(t)^{\lambda}
\]
holds for all $s, t \in I, \lambda \in [0, 1]$.

**Remark 5.** If a function is continuous and log-convex in the Jensen sense then it is also log-convex. We can also easily see that for positive functions exponential convexity implies log-convexity (consider the Definition 3 for $n = 2$).

The following two lemmas are equivalent to the definition of convexity (see [12, page 2]).

**Lemma 1.** Let $f : I \to \mathbb{R}$ be a convex function. Then for any $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ the following is valid
\[
(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0.
\]

**Lemma 2.** Let $f : I \to \mathbb{R}$ be a convex function. Then for any $x_1, x_2, y_1, y_2, \in I$ such that $x_1 \leq y_1, x_2 \leq y_2, x_1 \neq x_2, y_1 \neq y_2$ the following is valid
\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.
\]

In order to obtain results regarding the exponential convexity, we define the families of functions as follows.

For every choice of $l + 1$ mutually different points $x_0, x_1, ..., x_l \in [\alpha, \beta]$ we define

- $\mathcal{F}_1 = \{\phi_t : [\alpha, \beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is $n$-exponentially convex in the Jensen sense on $I\}$
- $\mathcal{F}_2 = \{\phi_t : [\alpha, \beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is exponentially convex in the Jensen sense on $I\}$
- $\mathcal{F}_3 = \{\phi_t : [\alpha, \beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is $2$-exponentially convex in the Jensen sense on $I\}$

**Theorem 16.** Let $F^H$ be the linear functional defined as in (4.1) associated with family $\mathcal{F}_1$. Then the following statements hold:

(i) The function $t \mapsto F^H(\phi_t)$ is $n$-exponentially convex in the Jensen sense on $I$.

(ii) If the function $t \mapsto F^H(\phi_t)$ is continuous on $I$, then it is $n$-exponentially convex on $I$.

**Proof.** (i) Let $p_j, s_j \in \mathbb{R}, j = 1, ..., n,$ and $s_{jk} = \frac{s_i + s_k}{2}, 1 \leq j, k \leq n$. We consider the function $h : [\alpha, \beta] \to \mathbb{R}$ defined by
\[
h(x) = \sum_{j,k=1}^{n} p_j p_k \phi_{s_{jk}}(x),
\]
where $\phi_{s_{jk}} \in \mathcal{F}_1$.

Since the function $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is $n$-exponentially convex in the Jensen sense on $I$, we have
\[
[x_0, x_1, ..., x_l; h] = \sum_{j,k=1}^{n} p_j p_k [x_0, x_1, ..., x_l; \phi_{s_{jk}}] \geq 0.
\]
Hence, the function $h$ is $l$-convex. Therefore, we have

$$F^H(h) = \sum_{j,k=1}^{n} p_j p_k F^H(\phi_{s_{jk}}) \geq 0.$$ 

Now we conclude that the function $t \mapsto F^H(\phi_t)$ is $n$-exponentially convex in the Jensen sense on $I$ what we need to prove.

(ii) Follows from (i) and Definition 3. $\square$

The following corollary is an easy consequence of the previous theorem.

**Corollary 1.** Let $F^H$ be the linear functional defined as in (4.1) associated with family $F_3$. Then the following statements hold:

(i) The function $t \mapsto F^H(\phi_t)$ is exponentially convex in the Jensen sense on $I$.

(ii) If the function $t \mapsto F^H(\phi_t)$ is continuous on $I$, then it is exponentially convex on $I$.

**Corollary 2.** Let $F^H$ be the linear functional defined as in (4.1) associated with family $F_3$. Then the following statements hold:

(i) If the function $t \mapsto F^H(\phi_t)$ is continuous on $I$, then it is 2-exponentially convex on $I$. If $t \mapsto F^H(\phi_t)$ is additionally positive, then it is also log-convex on $I$. Furthermore, for every choice $r, s, t \in I$, such that $r < s < t$, it holds

$$[F^H(\phi_s)]^{t-r} \leq [F^H(\phi_r)]^{t-s} [F^H(\phi_t)]^{s-r}.$$ 

(ii) If the function $t \mapsto F^H(\phi_t)$ is positive and differentiable on $I$, then for all $r, s, u, v \in I$ such that $r \leq u, s \leq v$, we have

$$M_{r,s}(F^H, F_3) \leq M_{u,v}(F^H, F_3),$$

where

$$M_{r,s}(F^H, F_3) = \begin{cases} \left(\frac{F^H(\phi_r)}{F^H(\phi_s)}\right)^{\frac{1}{t-s}}, & r \neq s, \\ \exp\left(\frac{d}{dt}(F^H(\phi_r))\right), & r = s. \end{cases}$$

**Proof.** (i) The first part of statement is an easy consequence of Theorem 16 and the second one of Remark 5.

Since the function $t \mapsto F^H(\phi_t)$ is log-convex on $I$, i.e. the function $t \mapsto \log F^H(\phi_t)$ is convex on $I$, then applying Lemma 1 we have

$$(r-t)\log F^H(\phi_s) + (t-s)\log F^H(\phi_r) + (s-r)\log F^H(\phi_t) \geq 0$$

for every choice $r, s, t \in I$, such that $r < s < t$. Therefore, we have

$$[F^H(\phi_s)]^{t-r} \leq [F^H(\phi_r)]^{t-s} [F^H(\phi_t)]^{s-r}.$$ 

(ii) Applying Lemma 2 to the convex function $t \mapsto \log F^H(\phi_t)$, we get

$$\frac{\log F^H(\phi_r) - \log F^H(\phi_s)}{r-s} \leq \frac{\log F^H(\phi_u) - \log F^H(\phi_v)}{u-v}$$

for $r \leq u, s \leq v, r \neq u, s \neq v$. Therefore, we have

$$M_{r,s}(F^H, F_3) \leq M_{u,v}(F^H, F_3).$$

Case $r = s, u = v$ follows from (4.6) as limiting case. $\square$
Remark 6. Note that, with assumption that the functions from $F_1$, $F_2$, $F_3$ are differentiable, the results from Theorem 16, Corollary 1 and Corollary 2 still hold when two of the points $x_0, x_1, \ldots, x_l \in [\alpha, \beta]$ coincide. Further, if the functions from $F_1$, $F_2$, $F_3$ are $l$-times differentiable, the results still hold when all points coincide. These can be easily proved using (1.8) and some fact of the exponential convexity.

5. Examples

Using few families of convex functions which are given below, we construct different examples of exponentially convex functions. As consequences, applying the Cauchy type of mean-value theorem from the previous section to these special families of functions, we establish new classes of two-parameter Cauchy-type means that are symmetric and have monotone properties over both parameters.

Thought this section we denote

$$A_k = \sum_{p=1}^{l} u_p (x_p - \alpha)^{k+2} - \sum_{q=1}^{m} v_q (y_q - \alpha)^{k+2}$$

$$B_k = \sum_{p=1}^{l} u_p (x_p - \beta)^{k+2} - \sum_{q=1}^{m} v_q (y_q - \beta)^{k+2}$$

Example 3. Let us consider a family of functions

$$\Omega_1 = \{ \phi_t : \mathbb{R} \to \mathbb{R} : t \in \mathbb{R} \}$$

defined by

$$\phi_t(x) = \begin{cases} \frac{e^{tx}}{x^n}, & t \neq 0, \\ \frac{e^{tx}}{x^m}, & t = 0. \end{cases}$$

Since $d^n \phi_t(x) = e^{tx} > 0$, the function $\phi_t$ is $n$-convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \mapsto d^n \phi_t(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 16 we also have that $t \mapsto [x_0, \ldots, x_l; \phi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 1 we conclude that $t \mapsto F^H(\phi_t)$, is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although the mapping $t \mapsto \phi_t$ is not continuous for $t = 0$), so it is exponentially convex. For this family of functions, $\mu_{\eta, \zeta}(F^H, \Omega_1)$, from (4.5), becomes

$$\mu_{\eta, \zeta} = \left( \frac{F^H(\phi_\eta)}{F^H(\phi_\zeta)} \right)^{\frac{1}{\eta - \zeta}}, \eta \neq \zeta,$$

$$\mu_{\eta, \zeta} = \left( \frac{1}{\eta^n} \sum_{p=1}^{l} u_p e^{\eta x_p} - \sum_{q=1}^{k} v_q e^{\eta y_q} - \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{\eta^{k+1}}{k!} [e^{\alpha \eta} A_k - e^{\beta \eta} B_k] \right)^{\frac{1}{\eta - \zeta}}, \eta \neq \zeta, \eta, \zeta \neq 0.$$
\[ \mu_{\eta, \eta} = \exp \left( \frac{\sum_{p=1}^{l} u_p x_p e^{\eta x_p} - \sum_{q=1}^{k} v_q y_q e^{\eta y_q} - K_1}{\sum_{p=1}^{l} u_p e^{\eta x_p} - \sum_{q=1}^{k} v_q e^{\eta y_q} - K_2} - \frac{n}{\eta} \right), \eta \neq 0. \]

\[ \mu_{0,0} = \exp \left( \frac{\sum_{p=1}^{l} u_p x_p^{n+1} - \sum_{q=1}^{k} v_q y_q^{n+1} - K_3}{n + 1} \right), \]

where

\[ K_1 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} x_k^{(k+2)} \left[ x^{\alpha k} e^{\eta x} \right]_{x=\alpha} A_k + \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} x_k^{(k+2)} \left[ x^{\beta k} e^{\eta x} \right]_{x=\beta} B_k, \]

\[ K_2 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} x_k^{(k+2)} \left[ e^{\alpha k} A_k - e^{\beta k} B_k \right], \]

\[ K_3 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \sum_{\eta=0}^{n-k} (n-\eta-1)! (n-k-\eta)! \left[ \frac{d^{n-k} \phi_\eta}{dx^{n-k}} \right] |_{x=\alpha} A_k - \beta^{n-k} B_k, \]

\[ K_4 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \sum_{\eta=0}^{n-k} (n-\eta-1)! (n-k-\eta)! \left[ \frac{d^{n-k} \phi_\eta}{dx^{n-k}} \right] |_{x=\beta} B_k. \]

By Corollary 2, \( \mu_{\eta, \zeta}(F^H, \Omega_1) \) is a monotonic mean in parameters \( \eta \) and \( \zeta \).

Since

\[ \left( \frac{d^n \phi_\eta}{dx^n} \right) \left( \frac{d^n \phi_\zeta}{dx^n} \right) \frac{1}{n!} (\log x) = x, \]

using Theorem 14 it follows that:

\[ M_{\eta, \zeta}(F^H, \Omega_1) = \log \mu_{\eta, \zeta}(F^H, \Omega_1). \]

satisfies

\[ \alpha \leq M_{\eta, \zeta}(F^H, \Omega_1) \leq \beta. \]

So, \( M_{\eta, \zeta}(F^H, \Omega_1) \) is a monotonic mean.

**Example 4.** Let us consider a family of functions

\[ \Omega_2 = \{ g_t : (0, \infty) \to \mathbb{R} : t \in \mathbb{R} \} \]

defined by

\[ g_t(x) = \begin{cases} x^{t-1} \log x, & t \notin \{0, 1, \ldots, n-1\}, \\ x^{t-1} \log x, & t = j \in \{0, 1, \ldots, n-1\}. \end{cases} \]

Since \( \frac{d^n g_t}{dx^n}(x) = x^{t-n} > 0 \), the function \( g_t \) is \( n \)-convex for \( x > 0 \) and \( t \mapsto \frac{d^n g_t}{dx^n}(x) \) is exponentially convex by definition. Arguing as in Example 3 we get that the mapping \( t \mapsto F^H(g_t) \) is exponentially convex. Hence, for this family of functions \( \mu_{\eta, \zeta}(F^H, \Omega_2) \), from (4.5), is equal to

\[ \mu_{p,q}(F^H, \Omega_2) = \begin{cases} \left( \frac{F^H(g_p)}{F^H(g_q)} \right)^{\frac{p-q}{n}}, & p \neq q, \\ \exp \left( \frac{(-1)^{n-1}(n-1)!}{2} F^H(g_p) + \sum_{k=0}^{n-1} \frac{1}{k-p} \right), & p = q \notin \{0, 1, \ldots, n-1\}, \\ \exp \left( \frac{(-1)^{n-1}(n-1)!}{2} F^H(g_p) + \sum_{k=0}^{n-1} \frac{1}{k-p} \right), & p = q \in \{0, 1, \ldots, n-1\}. \end{cases} \]
where 

\[ \mu_{\eta, \zeta} = \left( \frac{\zeta(\zeta - 1) \ldots (\zeta - n + 1)}{(\eta - 1) \ldots (\eta - n + 1)} \right) \]

\[ \times \left( \sum_{p=1}^{l} u_p x_p^\eta - \sum_{q=1}^{k} v_q y_q^\eta - \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{\eta(\eta - 1) \ldots (\eta - k + 2)}{k!} \left[ \alpha^{n-k-1} A_k - \beta^{n-k-1} B_k \right] \right)^{\frac{1}{\eta - \zeta}} \]

\[ \sum_{p=1}^{l} u_p x_p^\zeta - \sum_{q=1}^{k} v_q y_q^\zeta - \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{\zeta(\zeta - 1) \ldots (\zeta - k + 2)}{k!} \left[ \alpha^{n-k-1} A_k - \beta^{n-k-1} B_k \right] \]

\[ \eta \neq \zeta, \eta \notin \{0, 1, \ldots, n - 1\} \]

\[ \mu_{\eta, \eta} = \exp \left( \frac{\sum_{p=1}^{l} u_p x_p^\eta \log x_p - \sum_{q=1}^{k} v_q y_q^\eta \log y_q - C_1}{\sum_{p=1}^{l} u_p x_p^\eta - \sum_{q=1}^{k} v_q y_q^\eta - D_1} + \sum_{k=0}^{n-1} \frac{1}{k - \eta} \right), \]

\[ \eta \notin \{0, 1, \ldots, n - 1\} \]

\[ \mu_{\eta, \eta} = \exp \left( \frac{\sum_{p=1}^{l} u_p x_p^\eta \log^2 x_p - \sum_{q=1}^{k} v_q y_q^\eta \log^2 y_q - C_2}{2 \left( \sum_{p=1}^{l} u_p x_p^\eta \log x_p - \sum_{q=1}^{k} v_q y_q^\eta \log y_q - C_1 \right)} + \sum_{k=0}^{n-1} \frac{1}{k - \eta} \right), \]

\[ \eta \notin \{0, 1, \ldots, n - 1\} \]

where \( C_1 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{\eta(\eta - 1) \ldots (\eta - k + 2)}{k!} \left[ A_k \frac{d^{k+1}}{dx^{k+1}} (x^\eta \log x)|_{x=\alpha} - B_k \frac{d^{k+1}}{dx^{k+1}} (x^\eta \log x)|_{x=\beta} \right], \)

\[ D_1 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{\eta(\eta - 1) \ldots (\eta - k + 2)}{k!} \left[ \alpha^{n-k-1} A_k - \beta^{n-k-1} B_k \right], \]

\[ C_2 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k!} \left[ A_k \frac{d^{k+1}}{dx^{k+1}} (x^\eta \log^2 x)|_{x=\alpha} - B_k \frac{d^{k+1}}{dx^{k+1}} (x^\eta \log^2 x)|_{x=\beta} \right] \]

and

Again, using Theorem 14 we conclude that

\[ \alpha \leq \left( \frac{F_H(g_\eta)}{F_H(g_\zeta)} \right)^{\frac{1}{\eta - \zeta}} \leq \beta. \]

So, \( \mu_{\eta, \zeta}(F^H, \Omega_2) \), is a mean and by (4.4) it is monotonic.

**Example 5.** Let

\[ \Omega_3 = \{ \psi_t : (0, \infty) \to (0, \infty) : t \in (0, \infty) \} \]

be a family of functions defined by

\[ \psi_t(x) = \begin{cases} \frac{t^{-x}}{(\log t)^n}, & t \neq 1; \\ \frac{x^n}{n!}, & t = 1. \end{cases} \]

Since \( \frac{d^n}{dx^n}(x) = t^{-x} \) is the Laplace transform of a non-negative function (see [15]) it is exponentially convex. Obviously \( \psi_t \) are n-convex functions for every \( t > 0 \).
For this family of functions, $\mu_{\eta, \zeta} \left( F^H, \Omega_3 \right)$, in this case $[\alpha, \beta] \subseteq \mathbb{R}^+$, from (4.5) becomes

$$
\mu_{\eta, \zeta} \left( F^H, \Omega_3 \right) = \left( \frac{F^H(\eta)}{F^H(\zeta)} \right)^{\frac{1}{\eta - \zeta}}, \eta \neq \zeta;
$$

$$
\mu_{\eta, \eta} = \exp \left( \frac{\sum_{q=1}^{k} v_q y_q^{-\eta} - \sum_{p=1}^{l} u_p x_p \eta^{-x_p} + C_2}{\sum_{p=1}^{l} u_p \eta^{1-x_p} - \sum_{q=1}^{k} v_q \eta^{-y_q} - D_2} - \frac{n}{\eta \log \eta} \right), \eta \neq 1,
$$

$$
\mu_{1,1} = \exp \left( -\frac{1}{n+1} \sum_{p=1}^{l} u_p x_p^{n+1} - \sum_{q=1}^{k} v_q y_q^{n+1} - C_3 \right),
$$

where $C_1 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k! (k+2)} \left[ A_k (-1)^{k+1} \eta^{-\alpha} (\log \eta)^{k+1} - B_k (-1)^{k+1} \eta^{-\beta} (\log \eta)^{k+1} \right]$, $D_1 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k! (k+2)} \left[ A_k (-1)^{k+1} \zeta^{-\alpha} (\log \zeta)^{k+1} - B_k (-1)^{k+1} \zeta^{-\beta} (\log \zeta)^{k+1} \right]$, $C_2 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k! (k+2)} \left[ A_k (x\eta^{-x}) \vert_{x=\alpha} - B_k d^k d^{k+1} \frac{d}{dx} (x\eta^{-x}) \vert_{x=\beta} \right]$, $D_2 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k! (k+2)} \left[ A_k (-1)^{k+1} \eta^{-\alpha} (\log \eta)^{k+1} - B_k (-1)^{k+1} \eta^{-\beta} (\log \eta)^{k+1} \right]$, $C_3 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k! (k+2)} \left[ A_k (n-k) \beta^{n-k} - B_k (n-k) \beta^{n-k} \right]$ and $D_3 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k! (k+2)} \left[ A_k (n-k) \beta^{n-k} - B_k (n-k) \beta^{n-k} \right]$.

By (4.4) $\mu_{\eta, \zeta}$ is monotonic function in parameters $\eta$ and $\zeta$.

Using Theorem 14 it follows that

$$
M_{\eta, \zeta} \left( F^H, \Omega_3 \right) = -L(\eta, \zeta) \log \mu_{\eta, \zeta} \left( F^H, \Omega_3 \right),
$$

satisfy

$$
\alpha \leq M_{\eta, q} \left( F^H, \Omega_3 \right) \leq \beta.
$$

This shows that $M_{\eta, q} \left( F^H, \Omega_3 \right)$ is mean. Because of the above inequality (4.4), this mean is also monotonic. $L(\eta, q)$ is logarithmic mean defined by

$$
L(\eta, q) = \begin{cases} 
\frac{\eta - q}{\log \eta - \log q}, & \eta \neq q; \\
\eta, & \eta = q.
\end{cases}
$$

**Example 6.** Let

$$
\Omega_4 = \{ \gamma_t : (0, \infty) \to (0, \infty) : t \in (0, \infty) \}
$$

be a family of functions defined by

$$
\gamma_t(x) = e^{-x \sqrt{t}} / (-\sqrt{t})^\frac{n}{2}.
$$
Since \( \frac{d^n}{dx^n} (x) = e^{-x\sqrt{T}} \) is the Laplace transform of a non-negative function (see [15]) it is exponentially convex. Obviously \( \gamma_t \) are n-convex function for every \( t > 0 \).

For this family of functions, \( \mu_{\eta, \zeta} (F^H, \Omega_4) \), in this case for \([\alpha, \beta] \in \mathbb{R}^+ \), from (4.5) becomes

\[
\mu_{\eta, \zeta} = \left( \sum_{p=1}^{l} u_p e^{-x_p \sqrt{T}} - \sum_{q=1}^{k} v_q e^{-y_q \sqrt{T}} - C_1 \right)^{\frac{1}{n-\alpha}}, \eta \neq \zeta
\]

\[
\mu_{\eta, \eta} = \exp \left( \frac{\sum_{q=1}^{k} v_q y_q e^{-y_q \sqrt{T}} - \sum_{p=1}^{l} u_p x_p e^{-x_p \sqrt{T}} + C_2}{2\sqrt{\eta} \left( \frac{\sum_{p=1}^{l} u_p e^{-x_p \sqrt{T}} - \sum_{q=1}^{k} v_q e^{-y_q \sqrt{T}} - C_1}{\eta} \right) - \eta} \right)
\]

where \( C_1 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k!(k+2)} \left[ A_k (-1)^k \frac{k+1}{2} e^{-\alpha \sqrt{T}} - B_k (-1)^{k+1} \frac{k+1}{2} e^{-\beta \sqrt{T}} \right] \),

\( D_1 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k!(k+2)} \left[ A_k (-1)^k \frac{k+1}{2} e^{-\alpha \sqrt{T}} - B_k (-1)^{k+1} \frac{k+1}{2} e^{-\beta \sqrt{T}} \right] \),

\( C_2 = \frac{1}{\beta - \alpha} \sum_{k=0}^{n-2} \frac{1}{k!(k+2)} \left[ A_k \frac{k+1}{2} (xe^{-x \sqrt{T}}) |_{x=\alpha} - B_k \frac{k+1}{2} (xe^{-x \sqrt{T}}) |_{x=\beta} \right] \).

By (4.4) \( \mu_{\eta, \zeta} \) is monotonic function in parameters \( \eta \) and \( \zeta \).

Using Theorem 14 it follows that

\[
M_{\eta, \zeta} (F^H, \Omega_4) = - \left( \sqrt{\eta} + \sqrt{\zeta} \right) \log \mu_{\eta, \zeta} (F^H, \Omega_4)
\]

satisfy

\[
\alpha \leq M_{\eta, \zeta} (F^H, \Omega_4) \leq \beta.
\]

This shows that \( M_{\eta, \zeta} (F^H, \Omega_4) \) is mean. Because of the above inequality (4.4), this mean is also monotonic.

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STUDY OF SAMPLES OF PIPES MADE OF LAYERED COMPOSITE MATERIALS WITH TECHNOLOGICAL DEFECTS

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Abstract: We presents the results of research of samples for pipes made of composite materials with the technological defects.

Keywords: the technological defects, composite materials, simulation of deformation.

The use of layered composite materials in the modern systems of communication will allow us to solve problems of corrosion stability of the pipe, reduction of factor of linear expansion, salt deposits and paraffins, the reduction of hydraulic losses in the pipes, the cheapening of the cost of construction - installation works and the cost of maintenance and protection of the pipelines. In addition, the use of pipes made of layered composite materials instead of conventional metal, increases the service life of pipelines in 5-8 times. High specific indicators of the strength and rigidity of layered composite materials, along with chemical resistant, relatively small weight and other features that have made these materials attractive for the manufacture of pipelines of various purposes. However, due to the imperfection of technology, in structures made of laminated composites occur via defects of various shape and depth. Such defects can occur not only in manufacturing, but also the exploitation, transportation and storage. One of the features of the process of separation is the local buckling and buckling layer peeled off with the subsequent growth of the defect. In the work to confirm analytical dependencies and computer modeling, obtained by the authors in [1, 2, 3], were the samples were tested with defects of the type of delaminations, the methodology of the experiment. Testing of pipes of layered composite materials with technological defects is one of the useful tasks.
Realization of modeling and experimental works

In accordance with the developed methodology experimental works were previously simulated on samples of layered composite materials with application of integrated CAD/CAE systems. In the system ANSYS model was created ring, consisting of two thin volume of the shells in the form of rings using geometric primitives of layered composite material, one of which simulates delamination layer, the second - the rest of the design. The model was divided into a grid of finite elements when using the SOLID92. Problem of interaction of the bulk of shells was solved in the form of a contact problem of the two deformable bodies, was used for this contact elements from the library ANSYS. Embodied technological defects were simulated unmarked nodes elements of the contacting surfaces. The investigated model the ring has the axis of symmetry, so modeled ½ of the ring, taking into account that the planes of symmetry of the corresponding movement of zero. Model of ring loaded uniform external pressure, according to Fig. 1.

![Fig. 1. Ring, loaded with external pressure, with a defect type of peel, where D is the outer diameter of the ring, Q - uniform pressure.](image)

Samples were made of prepregs fiberglass industrial brands way to dry the computations. For this purpose in forming tools (mold), in the form of a steel ring of different inner diameter, blurred coating, laid revealed prepreg in five or ten layers. Then at the heart of the rings was put silicone insert to create the necessary internal pressure. With the ends of the mold is closed metal plates, after which the Assembly mounted clamp and place in the oven for polymerization (Fig. 2, 3).
Fig. 2. Stress state of the rings with a defect, the laying of prepreg [90,0]5 loaded pressure q=100H/mm². Stress plot: qkp=69H/mm²

Fig. 3. Stress state of the rings with a defect, the laying of prepreg [45,-45]5 loaded pressure q=100H/mm². Stress plot: qkp=27H/mm²

Were examined samples of pipes of layered composite materials with different diameters, the angles of prepregs and the depth of defects (Fig. 4).

Fig. 4. Sample of fiberglass D = 90 mm, laying prepreg [90,0]5, defect - 1200
Samples were made from prepregs fiber glass fabrics of industrial marks by way of the dry distribution. For this purpose in this instrumental tooling (mould), in the form of a steel ring of the different internal diameter greased by an antiadhesive covering, gave all the best preliminary the cut out prepreg in five or ten layers. Then in a core of a ring it was put silicone the loose leaf for creation of necessary internal pressure. From end faces прессформа it was closed by metal plates then assembly pulled together and placed in the furnace for polymerization. Defects, to simulate the interlaminar delaminations, were created by laying of a фторопластовый film between certain layers. After thermostabilization samples were exposed to machining from end faces. Border defects on samples outlines the risks.

Two options have been designed and manufactured for model loading devices and clamps to ensure full squeeze pressure. The first version of the useful model of device used in experiments for samples with diameter 90 mm (figure 5, a), second - for samples with diameter 50 mm (figure 5, b).

![A - a Useful model (1 variant) b - a Useful model (2 variant)](image)

Fig. 5. Useful model loading devices and terminals.

Useful model loading devices and terminals were put into a discontinuity machine with a capacity of 5 tons (Fig. 6), then the Assembly has been subjected to static efforts in tension and compression, the testimony of which were recorded with a digital recording device bursting of the machine and taking on the camcorder №1.
Test samples were produced in the form of a ring with a stuffy technological defect, in determining in each of the series of average critical force prior to the loss of stability - of defect type of peel. Scheme of loading of a utility model (1 option), is shown in Fig. 7. The pictures were made of the samples before and after the experiment; loading process, deformation and destruction was recorded on a video camera №2 (Fig. 8, 9). The results of experimental tests are presented in tables 1, 2, 3. Experimental values of the critical power loss of stability of the samples are shown in table 4, made numerical calculations in the system ANSYS - PANSYS.
Fig. 9. Sample of fiberglass after the loss of stability.

Table 1
The results of the experiment, produced by useful model (1 variant)

<table>
<thead>
<tr>
<th>№ exp.</th>
<th>№ sample</th>
<th>amount of the layers</th>
<th>Amount of the layers d/defect</th>
<th>Stowage prepreg</th>
<th>material</th>
<th>diameter sample, mm</th>
<th>defect of the type delamination</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>presence delamination</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>90,0</td>
<td>T-10</td>
<td>90</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>10</td>
<td>1</td>
<td>90,0</td>
<td>T-10</td>
<td>90</td>
<td>+</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>10</td>
<td>1</td>
<td>45,-45</td>
<td>T-10</td>
<td>90</td>
<td>+</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>10</td>
<td>1</td>
<td>90,0</td>
<td>T-10</td>
<td>90</td>
<td>+</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>10</td>
<td>1</td>
<td>45,-45</td>
<td>T-10</td>
<td>90</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2
The results of the experiment, produced by useful model (2 variant)

<table>
<thead>
<tr>
<th>№ exp.</th>
<th>№ sample</th>
<th>amount of the layers</th>
<th>Amount of the layers d/defect</th>
<th>Stowage prepreg</th>
<th>material</th>
<th>diameter sample, mm</th>
<th>defect of the type delamination</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>presence delamination</td>
</tr>
<tr>
<td>6</td>
<td>17</td>
<td>5</td>
<td>1</td>
<td>45,-45</td>
<td>ПС-ИФ</td>
<td>50</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>5</td>
<td>1</td>
<td>45,-45</td>
<td>ПС-ИФ</td>
<td>50</td>
<td>+</td>
</tr>
<tr>
<td>8</td>
<td>19</td>
<td>5</td>
<td>1</td>
<td>45,-45</td>
<td>ПС-ИФ</td>
<td>50</td>
<td>+</td>
</tr>
</tbody>
</table>
### Summary table of the results of the experiment

<table>
<thead>
<tr>
<th>№ exp.</th>
<th>№ sample</th>
<th>amount of the layers</th>
<th>Amount of the layers d/defect</th>
<th>Stowage prepreg</th>
<th>material</th>
<th>diameter sample, mm</th>
<th>defect of the type delamination</th>
<th>presence delamination</th>
<th>P(кН)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>90,0</td>
<td>T-10</td>
<td>90</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td>90,0</td>
<td>T-10</td>
<td></td>
<td>+</td>
<td>90</td>
<td>20,64</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td></td>
<td></td>
<td>45,-45</td>
<td>T-10</td>
<td></td>
<td>+</td>
<td>90</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td></td>
<td></td>
<td>90,0</td>
<td>T-10</td>
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<td>+</td>
<td>45</td>
<td>17,88</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td></td>
<td></td>
<td>45,-45</td>
<td>T-10</td>
<td></td>
<td>-</td>
<td>45</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
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<td>-</td>
<td>45</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td></td>
<td></td>
<td>45,-45</td>
<td>ПС-ИФ</td>
<td></td>
<td>+</td>
<td>45</td>
<td>4,5</td>
</tr>
</tbody>
</table>
## The results of calculations and tests

<table>
<thead>
<tr>
<th>amount of the layers</th>
<th>Amount of the layers d/defect type of the defect</th>
<th>Stowage prepreg diameter sample, mm</th>
<th>critical power of the loss to stability defect of the type delamination (kH)</th>
<th>Recsper</th>
<th>PANSYS*</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
<td>90,0</td>
<td>20,64</td>
<td>19, 36</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>45,-45</td>
<td></td>
<td>10</td>
<td>8,72</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>45,-45</td>
<td>4,5</td>
<td>4,9</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>90,0</td>
<td>8,71</td>
<td>7,19</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>90,0</td>
<td>20,3</td>
<td>21, 69</td>
<td></td>
</tr>
</tbody>
</table>

Table 4
- for comparison with the experimental data, uniform pressure acting on the ring-road, the numerical calculation, translated in load effort.

According to the results of the experimental studies were established the following dependencies:

- in 5 - and 10-layer samples, with the same characteristics, with its technological defect at a depth of one layer, when laying reinforced tissue under the angle \([45,-45]\) critical force for stability loss of a defect-type peeling reduced by 50% in contrast to the laying of reinforced tissue under the angle \([0, 90]\) (table. 1, 2);
- diameters samples of pipes has no significant impact on the critical force of the stability loss of a defect-type peeling (table. 3).

The results obtained in the work

1. Developed methodology of the experiment, carried out a series of tests. The results of experimental works are presented in the form of tables.
2. The method was developed for the simulation of elements of structures with defects located at a different depth in the system of ANSYS.
3. Made numerical modeling and strength calculation by the method of final elements in the system of ANSYS. The results of the calculation are presented in the form of a table. The results obtained in the course of the experiment is comparable with the numerical calculations, the degree of convergence in the limits of 15-20 per cent.
4. The effect of the angle of laying prepregs in the samples on the critical pressure and for critical behavior with the subsequent growing defect.
5. It is revealed, that the diameters of the tubes samples have little effect on the critical force of the stability loss of a defect-type peeling.

Bibliography:

SPECTRAL PROPERTIES OF ReV AND ImV

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Abstract. We studied spectrum, numerical range and operator norm of ReV and Im V on $L^2(0,1)$.

1. Introduction

For a bounded linear operator $T$ on a complex Hilbert space $H$, the numerical range $W(T)$ is the image of the unit sphere of $H$ under the quadratic form $x \to (Tx, x)$ associated with the operator. More precisely, $W(T) = \{ (Tx, x) : x \in H, ||x|| = 1 \}$.

It is well known (see for instance [2]) that:

- The numerical range of an operator is convex; (the Toeplitz- Hausdorff theorem)
- The spectrum of an operator is contained in the closure of its numerical range;
- $T$ is a self-adjoint if and only if $W(T) \subset \mathbb{R}$.

A bounded linear operator $T$ on $H$ is accretive, if

$$ReT = \frac{T + T^*}{2} \geq 0.$$ 

The resolvent set for $T$ is the set $\rho(T)$ consisting of all complex $\lambda$ such as $T - \lambda I$ is bijective and its inverse is continuous. (where $I$ is the identity operator $H$)

The $R(\lambda, T) = (T - \lambda I)^{-1}, \lambda \in \rho(T)$ is called the resolvent operator for $T$. The complement set $\sigma(T)$ of $\rho(T)$ called the spectrum of $T$, includes the following two by two disjoint sets:

- the point spectrum $\sigma_p(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is noninjective} \}$,
the continuous spectrum $\sigma_c(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is injective, nonsurjective} \}$ and $(T - \lambda I)H = H$,

- the residual spectrum $\sigma_r(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is injective} \}$ and $(T - \lambda I)H \neq H$.

We denote by $V$ the classical Volterra operator

$$(Vf)(x) = \int_0^x f(s)ds, \quad 0 < x < 1$$
on $L^2(0, 1)$.

We recall the well-known formula

$$(V^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x - s)^{n-1} f(s)ds$$

and

$$(V^*n f)(x) = \frac{1}{(n-1)!} \int_x^1 (s - x)^{n-1} f(s)ds, \quad 0 < x < 1, \ n \in \mathbb{N}.$$}

The adjoint of the Volterra operator is

$$(V^*f)(x) = \int_x^1 f(s)ds.$$"
For any compact linear operator $T$ in a complex Hilbert space the singular numbers $s_k(T)$ are the distances from $T$ to the set of all operators of rank less than or equal to $k - 1$, $k \geq 1$. Their squares are the eigenvalues of the compact self-adjoint nonnegative operator $T^*T$ counted according to multiplicities. (see [1]) In particular, $s_1(T) = \|T\|$. The Halmos calculation (see [3]) yields

$$s_k(V) = \frac{2}{(2k - 1)\pi}$$

for all $k \geq 1$, in particular

$$\|V\| = \frac{2}{\pi}$$

on $L^2(0,1)$.

We studied operator norm, spectrum and numerical range of Re$V$ and Im$V$ on $L^2(0,1)$.

2. The Results

**Proposition 2.1.** The operator norm of Re$V$ is $\frac{1}{2}$, that is,

$$\|\text{Re}V\| = \frac{1}{2}.$$

**Proof.** The point is that the spectral problem

$$(V^2 + (V^*)^2 + VV^* + V^*V)f(x) = 4\lambda f(x), \quad 0 \leq x \leq 1$$

can be rewritten as

$$\int_0^x (x-t)f(t)dt + \int_0^1 (t-x)f(t)dt + \int_0^x \left( \int_0^t f(\xi)d\xi \right)dt +$$

$$+ \int_0^x \left( \int_0^t f(\xi)d\xi \right)dt = 4\lambda f(x) \quad (1)$$

We proceed from this integral equation to a differential equation by applying the operator $D = \frac{d}{dx}$. Thus (1) yields $f(x) = c = \text{const}$.

Now we insert $f(x) = c$ into (1), we get $\lambda = \frac{1}{4}$. Therefore $\|\text{Re}V\| = \frac{1}{2}$. □

**Proposition 2.2.** The operator norm of Im$V$ is $\frac{1}{\pi}$, that is,

$$\|\text{Im}V\| = \frac{1}{\pi}.$$
Proof. The spectral problem

\[ ((V^2 + (V^*)^2 + VV^* + VV^*))f(x) = -4\lambda f(x), \quad 0 \leq x \leq 1 \]

can be rewritten as

\[
\int_0^x (x - t)f(t)dt + \int_x^1 (t - x)f(t)dt - \int_0^x \left( \int_0^t f(\xi)d\xi \right) dt + \\
- \int_x^1 \left( \int_0^t f(\xi)d\xi \right) dt = -4\lambda f(x).
\]  \( (2) \)

By applying the operator \( D = \frac{d}{dx} \) twice. Thus (2) yields

\[
\int_0^x f(t)dt - \int_x^1 f(t)dt = -2\lambda f'(x).
\]  \( (3) \)

and

\[
\lambda f'' + f(x) = 0.
\]  \( (4) \)

We insert \( x = 0 \) and \( x = 1 \) into (2) and (3), respectively. We obtain

\[
\begin{cases} 
\lambda f''(x) + f(x) = 0 \\
f(0) + f(1) = 0 \\
f'(0) + f'(1) = 0 
\end{cases}
\]  \( (5) \)

From (5) the corresponding singular number is

\[
\sqrt{\lambda_n} = \frac{1}{\pi(2n + 1)}, \quad n \in \mathbb{Z}_+.
\]

Therefore,

\[
||\text{Im}V|| = \frac{1}{\pi}.
\]

Proposition 2.3. The numerical range of \( \text{ReV} \) is \([0, \frac{1}{2}]\), that is,

\[
W(\text{ReV}) = [0, \frac{1}{2}].
\]
Proof. First, since $ReV$ is self-adjoint and bounded, $W(ReV)$ is a bounded subset of the real line. It is easy to see that

$((ReV) \cos \pi x, \cos \pi x) = 0, \quad ((ReV)1, 1) = \frac{1}{2}.$

For any function $f \in L^2[0, 1]$ the inequality $Re(Vf, f) \geq 0$ is satisfied, implying $0 \in \partial W(ReV)$ ($\partial W(ReV)$ is the boundary of $W(ReV)$).

By Proposition 1, yields $\frac{1}{2} \in \partial W(ReV)$, we obtain $W(ReV) = \left[0, \frac{1}{2}\right]$. □

Proposition 2.4. The numerical range of $ImV$ is $\left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$, that is,

$W(ImV) = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right].$

Proof. Since, $ImV$ is self-adjoint. By Proposition 2, $W(ImV)$ is a bounded convex subset of the $\left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$.

$Im(Ve^{-i\pi x}, e^{-i\pi x}) = \frac{1}{\pi}$ and $Im(Ve^{i\pi x}, e^{i\pi x}) = -\frac{1}{\pi}$.

Therefore, $W(ImV) = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$. □

Proposition 2.5. The $\sigma(ImV)$ has the following properties:

- $\sigma(ImV) = \left\{\frac{1}{\pi(2k + 1)} | k \in \mathbb{Z}\right\} \cup \{0\}$.
- $\sigma_p(ImV) = \left\{\frac{1}{\pi(2k + 1)} | k \in \mathbb{Z}\right\}$.
- $\sigma_c(ImV) = \{0\}$ and $\sigma_r(ImV) = \emptyset$.

Proposition 2.6. If $\lambda \in \rho(ImV)$, $\mu = \frac{1}{2\lambda}$ and $(Sf)(x) = e^{i\pi x}f(x)$ then

$R(\lambda, ImV) = -2\mu I - \frac{4\mu^2}{\cos \mu} S^{-1}(Im(e^{i\mu}V))S.$

Theorem 2.1. $\sigma(ReV) = \sigma_p(ReV) = \left\{0, \frac{1}{2}\right\}$ and $\sigma_c(ReV) = \sigma_r(ReV) = \emptyset$.

Proof. Consider the following inequality

$(ReV - \lambda I)f = \varphi$ (6)

for any $\varphi \in L^2[0, 1]$. 

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Denote by
\[ c = ((V + V^*)f)(x) = \int_0^1 f(x)dx \quad \text{and} \quad d = \int_0^1 \varphi(x)dx. \]

From (6), can be rewritten as
\[ \frac{c}{2} - \lambda f = \varphi. \]

Integrating by two sided, gives
\[ \left( \frac{1}{2} - \lambda \right) c = d. \]

We consider the following two cases.

1. If \( \lambda \neq 0, \lambda \neq \frac{1}{2} \) then \( c = \frac{2d}{1 - 2\lambda} \) implies that \( f = \frac{1}{\lambda} \left( \frac{d}{1 - 2\lambda} - \varphi \right) \).

By calculation, we have \( |d| \leq ||f|| \) and \( ||f|| \leq \left( \frac{1}{|\lambda(1 - 2\lambda)|} + \frac{1}{|\lambda|} \right) ||\varphi|| \), implies that \( f \in L^2[0, 1] \). Range and kernel of \( ReV - \lambda I \) are \( L^2(0, 1) \) and \( \{0\} \) respectively. Therefore, \( \lambda \in \rho(ReV) \).

2. \( (ReV - \frac{1}{2} \cdot I) \left( t - \frac{1}{2} \right) = 0 \) and \( (ReV - \frac{1}{2} \cdot I) 1 = 0 \) implies that \( 0, \frac{1}{2} \in \sigma_p(ReV) \).

\( \square \)

**Corollary 2.1.** If \( \lambda \in \rho(ReV) \), then \( R(\lambda, ReV) = \frac{1}{\lambda} \left( \frac{1}{2(1 - 2\lambda)} ReV - I \right) \).

**Acknowledgements.** The authors are grateful to Prof. Jaroslav Zemánek for useful discussions and to Institute of Mathematics of the Polish Academy of Sciences for its hospitality.

**References**


A Chi-Squared Test for Serial Randomness of Nominal Data with Multiple Categories

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March 14, 2015

Abstract
This test tests the randomness of a series of multiple nominal values. Construction of this test is based on statistical simulation and theoretical knowledge.

Keywords: nonparametric test, serial randomness, test for trend detection

1 Background
I have searched and studied tests for serial randomness (table 1 and figure 1). So I have concluded that need to develop a new test for serial randomness of nominal data with multiple categories.

<table>
<thead>
<tr>
<th>Test Variable Type</th>
<th>Category #</th>
<th>Usage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Run Test [3]</td>
<td>ordinal, nominal</td>
<td>2</td>
</tr>
<tr>
<td>2 Mann-Kendall’s Test [6]</td>
<td>ordinal</td>
<td>any</td>
</tr>
<tr>
<td>3 Bartels’ Rank Test [6]</td>
<td>ordinal</td>
<td>any</td>
</tr>
<tr>
<td>5 Gap Test [3]</td>
<td>ordinal, nominal (digits)</td>
<td>10 or any</td>
</tr>
</tbody>
</table>

Table 1: Existing tests for trend detection for categorical data

Figure 1: Test for serial randomness on SPSS, one of popular statistical packages.

(a) One sample tests
(b) Run Test
2 Null or alternative hypothesis

Let we have a sample $X = (X_1, X_2, \ldots, X_n)$. Where, $X_i \in \{a_1, a_2, \ldots, a_s\}$ - categorical (unordered) variable. Suppose $s \geq 3$. Let $H_0 : X$ random series $H_1 : X$ non-random series and it means

$H_0$ : no rule of “order” (no trend) and discrete uniform distributed

$H_1$ : has rule of “order” (trend) or non discrete uniform distributed.

3 Test statistics

Now we will give a statistics, this will be used to measure trend. Suppose

$$n_i = \sum_{j=1}^{n} I(X_j = a_i)$$

and

$$T_i = \sum_{j=1}^{n} I(X_j = a_i)j$$

where, $I(\cdot)$ is indicator function. Under the null hypothesis we have that $T_i/n_i$ has a normal distribution with expectation

$$\mu = E\left(\frac{T_i}{n_i}\right) = \frac{n+1}{2}$$

and variance $\sigma^2 = Var\left(\frac{T_i}{n_i}\right)$ by statistical simulation. Histogram of simulated value of random variable $T_i/n_i$ given in figure 2, when $s = 4$ and $i = 1$. Also result of normality test (One-Sample Kolmogorov-Smirnov Test [4, p.111]) given in table 2. Thus, for any $a_i$

$Z_i = \frac{T_i}{n_i} - \frac{n+1}{2} \sim N(0,1)$

Table 2: Result of normality test for $T_i/n_i$ when $s = 4$ and $i = 1$

<table>
<thead>
<tr>
<th>$i$</th>
<th>Estimated Mean</th>
<th>Estimated Std. Deviation</th>
<th>Most Extreme Difference</th>
<th>Kolmogorov-Smirnov Z</th>
<th>Asymp. Sig. (2-tailed)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100.15</td>
<td>7.28</td>
<td>0.029</td>
<td>0.660</td>
<td>0.777</td>
</tr>
<tr>
<td>2</td>
<td>100.13</td>
<td>7.29</td>
<td>-0.025</td>
<td>0.567</td>
<td>0.904</td>
</tr>
<tr>
<td>3</td>
<td>101.20</td>
<td>7.03</td>
<td>-0.028</td>
<td>0.636</td>
<td>0.814</td>
</tr>
<tr>
<td>4</td>
<td>100.59</td>
<td>7.19</td>
<td>0.022</td>
<td>0.493</td>
<td>0.968</td>
</tr>
</tbody>
</table>

Now we need to find estimation of standard deviation $\sigma$. Under the null hypothesis (discrete uniform distributed) $T_1/n_1, \ldots, T_s/n_s$ have same distribution (figure 3). Also we have show relation between number of categories, sample size and standard deviation of $T_i/n_i$ in the figure 4. Thus we can choose linear model.
Figure 3: Comparison between $T_1/n_1, \ldots, T_s/n_s$, when $s = 4$

Figure 4: Relation between number of categories, sample size and standard deviation of $T_i/n_i$
Moreover estimation of standard deviation was found as

$$\hat{\sigma} = 0.8s + 0.02n + 2$$

by linear regression analysis (table 3 and 4).

By residual analysis (figure 5 and table 5), we have concluded that (2) is linear model without heteroscedasticity [2]. Finally, by relation between normal distribution and chi-square distribution [5, p.193]

$$X^2 = \sum_{i=1}^{s} Z_i^2 \sim \chi^2(s-1)$$

### 4 Critical area and critical value

Under $H_0$ the asymptotic distribution of the test statistics is

$$X^2 \sim \chi^2(s-1)$$

which leads to the rejection region

$$X^2 = \sum_{i=1}^{s} \left( \frac{T_i}{n_i} - \frac{n + 1}{2} \right)^2 > \chi^2_{0.05, s-1}$$

where, $\chi^2_{0.05, s-1}$ - inverse value of the chi-square distribution with degrees of freedom $s$ for the corresponding probabilities in $1 - \alpha$.

### 5 Example

Let $X = (B, D, B, A, A, A, A, C, B)$. Observed value of test statistics:

$$X^2 = \left( \frac{22}{4} - 5 \right)^2 + \left( \frac{13}{3} - 5 \right)^2 + \left( \frac{8}{1} - 5 \right)^2 + \left( \frac{2}{1} - 5 \right)^2 \approx 9.81$$

Detailed calculation was given in table 6. Critical value at 0.05 level of significance is $\chi^2_{0.05, 3} = 7.81$.

$$X^2 = 9.81 > \chi^2_{0.05, 3} = 7.81$$

So we have concluded that the null hypothesis was rejected at 0.05 level of significance.

Table 5: Correlation between squared residual of model 2 and sample size, number of categories.
Figure 5: Relation between predicted standard deviation and squared residual of model (2)

<table>
<thead>
<tr>
<th>i</th>
<th>Category</th>
<th>(n_i)</th>
<th>(T_i)</th>
<th>(T_i/n_i)</th>
<th>(Z_i^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>4</td>
<td>4 + 5 + 6 + 7 = 22</td>
<td>5.5</td>
<td>0.131</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>3</td>
<td>1 + 3 + 9 = 13</td>
<td>4.3(3)</td>
<td>0.233</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>1</td>
<td>8</td>
<td>8</td>
<td>4.725</td>
</tr>
<tr>
<td>4</td>
<td>D</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4.725</td>
</tr>
</tbody>
</table>

\[X^2 = 9.811\]

Table 6: Test statistics calculation

6 Summary

This test is useful for large sample and suitable for categorical data with multiple categories. It requires further analysis to remove “discrete uniform distributed” from null hypothesis.

References


