A Note On Some New Inequalities Similar to Hilbert-Type inequality

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Abstract

In this paper, we establish some new inequalities similar to Hilbert-type integral inequality and the result of J. A. Oguntuase, L. -E. Persson and J. E. Pečarić [Aust. J. Math. Anal. Appl. **7(2)** (2011), Art. 19] with involving many functions.

1 Introduction

Oguntuase, Persson and Pečarić in [1] proved a new Hardy's inequality involving many functions. Their result is contained in the following theorem.

Theorem A. *Let $p > 0, p \neq 1$. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a positive sequence such $\sum_{k=1}^{\infty} \alpha_k = 1$ and $\{f_k\}_{k=1}^{\infty}$ be a sequence of integrable functions and let $F_k(x) = \int_0^x f_k(t)dt,\ k = 1, 2, \ldots$. Then the inequality*

$$\int_0^{\infty} \left(\prod_{k=1}^{\infty} \left[\frac{1}{x} F_k(x) \right]^{\alpha_k} \right)^p d \leq \left(\frac{p}{|p-1|} \right)^p \int_0^{\infty} \left(\sum_{k=1}^{\infty} \alpha_k f_k(x) \right)^p dx, \tag{1.1}$$

hold if and only if $p > 1$ and the constant factor $\left(\frac{p}{p-1}\right)^p$ is the best possible.

Theorem A is generalization of the result of L. Bougoffa [2, Theorem 2.2].

The main objective of this paper is to build some new inequalities similar to Hilbert-type integral inequalities and Hardy's inequality (1.1). Our results will be based on the following our result [3].

Theorem B. *Let p and q be conjugate parameters with $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \leq 1$, and let $\lambda, s, r > 0$ such that $s + r = \lambda$, $k_\lambda(x, y)$ is non-negative homogeneous function of degree $-\lambda$ in $\mathbb{R}_+^2$. Assume $F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt$.*

If

$$0 < \widetilde{C}_\lambda(s) < \infty, 0 < \int_0^{\infty} k_\lambda(1, u) u^{s - \frac{1}{p} - \beta} du < \infty, 0 < \int_0^{\infty} k_\lambda(1, u) u^{r - \frac{1}{q} - \alpha} du < \infty$$

and $f, g \geq 0$ satisfy

$$0 < \int_0^{\infty} f^{\alpha p}(x)dx < \infty, \quad 0 < \int_0^{\infty} g^{\beta q}(x)dx < \infty,$$

then the following two inequalities hold:

$$\int_0^{\infty} \int_0^{\infty} k_\lambda(x, y) x^{r - \frac{1}{q} - \alpha} y^{s - \frac{1}{p} - \beta} F^\alpha(x) G^\beta(y) dxdy$$
$$< C_\lambda(\alpha, \beta, s, p, q) \left\{ \int_0^{\infty} f^{\alpha p}(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\infty} g^{\beta q}(x)dx \right\}^{\frac{1}{q}} \tag{1.2}$$

and

$$\int_0^{\infty} \left(\int_0^{\infty} k_\lambda(x, y) x^{r - \frac{1}{q} - \alpha} y^{s - \frac{1}{p}} F^\alpha(x) dx \right)^p dy$$
$$< \widetilde{C}_\lambda^p(s) \left(\frac{\alpha p}{\alpha p - 1} \right)^{\alpha p} \int_0^{\infty} f^{\alpha p}(x)dx, \tag{1.3}$$

where $\widetilde{C}_\lambda(s) = \int_0^\infty k_\lambda(1,u)u^{s-1}du$ *and the constant factors*

$$C_\lambda(\alpha,\beta,s,p,q) = \widetilde{C}_\lambda(s)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta, \quad \widetilde{C}_\lambda^p(s)\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}$$

are the best possible.

2 Main Results

Theorem 2.1. *Let* p *and* q *be conjugate parameters with* $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \leq 1$, *and let* $\lambda, s, r > 0$ *such that* $s + r = \lambda$, $k_\lambda(x,y)$ *is non-negative homogeneous function of degree* $-\lambda$ *in* $\mathbb{R}_+^2$. *Let* $\{\alpha_k\}_{k=1}^\infty, \{\beta_k\}_{k=1}^\infty$ *be positive sequences such* $\sum_{k=1}^\infty \alpha_k = 1, \sum_{k=1}^\infty \beta_k = 1$ *and* $\{f_k\}_{k=1}^\infty, \{g_k\}_{k=1}^\infty$ *be sequences of non-negative integrable functions and let*

$$F_k(x) = \int_0^x f_k(t)dt, \quad G_k(y) = \int_0^y g_k(t)dt, \quad k = 1, 2, \ldots.$$

If

$$0 < \widetilde{C}_\lambda(s) < \infty, 0 < \int_0^\infty k_\lambda(1,u)u^{s-\frac{1}{p}-\beta}du < \infty, 0 < \int_0^\infty k_\lambda(1,u)u^{r-\frac{1}{q}-\alpha}du < \infty$$

and

$$0 < \int_0^\infty \left(\sum_{k=1}^\infty \alpha_k f_k(x)\right)^{\alpha p} dx < \infty, \quad 0 < \int_0^\infty \left(\sum_{k=1}^\infty \beta_k g_k(x)\right)^{\beta q} dx < \infty,$$

then the following two inequalities hold:

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)x^{r-\frac{1}{q}-\alpha}y^{s-\frac{1}{p}-\beta}\left(\prod_{k=1}^\infty F_k^{\alpha_k}(x)\right)^\alpha \left(\prod_{k=1}^\infty G_k^{\beta_k}(y)\right)^\beta dxdy$$
$$< C_\lambda(\alpha,\beta,s,p,q)\left\{\int_0^\infty \left(\sum_{k=1}^\infty \alpha_k f_k(x)\right)^{\alpha p} dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \left(\sum_{k=1}^\infty \beta_k g_k(x)\right)^{\beta q} dx\right\}^{\frac{1}{q}} \tag{2.1}$$

and

$$\int_0^\infty \left(\int_0^\infty k_\lambda(x,y)x^{r-\frac{1}{q}-\alpha}y^{s-\frac{1}{p}}\left(\prod_{k=1}^\infty F_k^{\alpha_k}(x)\right)^\alpha dx\right)^p dy$$
$$< \widetilde{C}_\lambda^p(s)\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}\int_0^\infty\left(\sum_{k=1}^\infty \alpha_k f_k(x)\right)^{\alpha p} dx, \quad (2.2)$$

where the constant factors

$$C_\lambda(\alpha,\beta,s,p,q) = \widetilde{C}_\lambda(s)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha\left(\frac{\beta q}{\beta q-1}\right)^\beta, \quad \widetilde{C}_\lambda^p(s)\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}$$

are the best possible.

Proof. According to the Arithmetic-Geometric Mean inequality

$$\prod_{k=1}^\infty h_k^{\alpha_k}(x) \le \sum_{k=1}^\infty \alpha_k h_k(x),$$

we have that

$$\left(\prod_{k=1}^\infty F_k^{\alpha_k}(x)\right)^\alpha \le \left(\sum_{k=1}^\infty \alpha_k F_k(x)\right)^\alpha = \left(\int_0^x\left(\sum_{k=1}^\infty \alpha_k f_k(t)\right)dt\right)^\alpha, \quad (2.3)$$

$$\left(\prod_{k=1}^\infty G_k^{\beta_k}(x)\right)^\beta \le \left(\sum_{k=1}^\infty \beta_k G_k(x)\right)^\beta = \left(\int_0^x\left(\sum_{k=1}^\infty \beta_k g_k(t)\right)dt\right)^\beta. \quad (2.4)$$

By using inequality (1.2) with the functions $\sum_{k=1}^\infty \alpha_k f_k(t)$, $\sum_{k=1}^\infty \beta_k g_k(t)$ and (2.3), (2.4) the inequality (2.1) is proved. The constant factor in the inequality (2.1) is the best possible since by applying it with $f_k(t) = f(t)$, $g_k(t) = g(t)$, $k = 1, 2, \ldots$, it reduces to (1.2) and it is known the constant factor in inequality (1.2) is best possible.

The proof of inequality (2.2) is similar to the proof of inequality (2.1). The theorem is proved. □

If $k_\lambda(x,y) = 1/(x+y)^\lambda$ or $1/\max\{x^\lambda, y^\lambda\}$ then we obtain the following corollaries correspondingly,

Corollary 2.2. *Let p and q be conjugate parameters with $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \leq 1$, and let $\lambda, s, r > 0$ such that $s + r = \lambda$. Let $\{\alpha_k\}_{k=1}^{\infty}, \{\beta_k\}_{k=1}^{\infty}$ be positive sequences such $\sum_{k=1}^{\infty} \alpha_k = 1, \sum_{k=1}^{\infty} \beta_k = 1$ and $\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty}$ be sequences of non-negative integrable functions and let*

$$F_k(x) = \int_0^x f_k(t)dt, \quad G_k(y) = \int_0^y g_k(t)dt, \quad k = 1, 2, \ldots.$$

If

$$0 < \int_0^\infty \left(\sum_{k=1}^{\infty} \alpha_k f_k(x) \right)^{\alpha p} dx < \infty, \quad 0 < \int_0^\infty \left(\sum_{k=1}^{\infty} \beta_k g_k(x) \right)^{\beta q} dx < \infty,$$

then the following two inequalities hold:

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}-\beta}}{(x+y)^\lambda} \left(\prod_{k=1}^{\infty} F_k^{\alpha_k}(x) \right)^{\alpha} \left(\prod_{k=1}^{\infty} G_k^{\beta_k}(y) \right)^{\beta} dxdy \\ < B(r,s) \left(\frac{\alpha p}{\alpha p - 1} \right)^{\alpha} \left(\frac{\beta q}{\beta q - 1} \right)^{\beta} \\ \times \left\{ \int_0^\infty \left(\sum_{k=1}^{\infty} \alpha_k f_k(x) \right)^{\alpha p} dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \left(\sum_{k=1}^{\infty} \beta_k g_k(x) \right)^{\beta q} dx \right\}^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} \int_0^\infty \left(\int_0^\infty \frac{x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}}}{(x+y)^\lambda} \left(\prod_{k=1}^{\infty} F_k^{\alpha_k}(x) \right)^{\alpha} dx \right)^{p} dy \\ < [B(r,s)]^p \left(\frac{\alpha p}{\alpha p - 1} \right)^{\alpha p} \int_0^\infty \left(\sum_{k=1}^{\infty} \alpha_k f_k(x) \right)^{\alpha p} dx, \end{aligned}$$

where the constant factors $B(r,s) \left(\frac{\alpha p}{\alpha p-1} \right)^{\alpha} \left(\frac{\beta q}{\beta q-1} \right)^{\beta}$ and $[B(r,s)]^p \left(\frac{\alpha p}{\alpha p-1} \right)^{\alpha p}$ are the best possible.

Corollary 2.3. *Let p and q be conjugate parameters with $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \leq 1$, and let $\lambda, s, r > 0$ such that $s + r = \lambda$. Let $\{\alpha_k\}_{k=1}^{\infty}, \{\beta_k\}_{k=1}^{\infty}$ be positive sequences* *such* $\sum_{k=1}^{\infty}\alpha_k = 1, \sum_{k=1}^{\infty}\beta_k = 1$ *and* $\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty}$ *be sequences of non-negative integrable functions and let*

$$F_k(x) = \int_0^x f_k(t)dt, \quad G_k(y) = \int_0^y g_k(t)dt, \quad k = 1, 2, \ldots.$$

If

$$0 < \int_0^{\infty}\left(\sum_{k=1}^{\infty}\alpha_k f_k(x)\right)^{\alpha p} dx < \infty, \quad 0 < \int_0^{\infty}\left(\sum_{k=1}^{\infty}\beta_k g_k(x)\right)^{\beta q} dx < \infty,$$

then the following two inequalities hold:

$$\int_0^{\infty}\int_0^{\infty}\frac{x^{r-\frac{1}{q}-\alpha}y^{s-\frac{1}{p}-\beta}}{\max\{x^{\lambda}, y^{\lambda}\}}\left(\prod_{k=1}^{\infty}F_k^{\alpha_k}(x)\right)^{\alpha}\left(\prod_{k=1}^{\infty}G_k^{\beta_k}(y)\right)^{\beta} dxdy$$
$$<\frac{\lambda}{rs}\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}\left(\frac{\beta q}{\beta q-1}\right)^{\beta}$$
$$\times\left\{\int_0^{\infty}\left(\sum_{k=1}^{\infty}\alpha_k f_k(x)\right)^{\alpha p} dx\right\}^{\frac{1}{p}}\left\{\int_0^{\infty}\left(\sum_{k=1}^{\infty}\beta_k g_k(x)\right)^{\beta q} dx\right\}^{\frac{1}{q}},$$

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{x^{r-\frac{1}{q}-\alpha}y^{s-\frac{1}{p}}}{\max\{x^{\lambda}, y^{\lambda}\}}\left(\prod_{k=1}^{\infty}F_k^{\alpha_k}(x)\right)^{\alpha} dx\right)^{p} dy$$
$$< \left(\frac{\lambda}{rs}\right)^{p}\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}\int_0^{\infty}\left(\sum_{k=1}^{\infty}\alpha_k f_k(x)\right)^{\alpha p} dx,$$

where the constant factors $\frac{\lambda}{rs}\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha}\left(\frac{\beta q}{\beta q-1}\right)^{\beta}$ *and* $\left(\frac{\lambda}{rs}\right)^{p}\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}$ *are the best possible.*

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