Comparison of method of successive approximations (MSA) for parabolic PDE's with Adomian decomposition method

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Abstract

The well known MSA is one of the powerful and popular methods in numerical analysis. In this paper, we have shown that MSA and Adomian's decomposition methods are identical for heat equation. We have used the MSA to solve the heat and Burger's equation. Some special cases of the equation are solved as examples to show the efficiency and ability of the method.

1 Introduction

The method of successive approximations is one of the powerful methods for solving operator equations:

$$F(u) = 0. \tag{1.1}$$

The fixed-point iteration consists of transforming this equation into an equivalent one

$$u = \Phi(u) \tag{1.2}$$

and of constructing a sequence $\{u_n\}$ with the help of the iteration scheme

$$u_{n+1} = \Phi(u_n), \qquad n = 0, 1, \ldots \tag{1.3}$$

where u_0 is a given starting value. The iteration (1.3) refereed as MSA. As is known in [1], the iterative process (1.3) is convergent, if the operator Φ is contractive mapping with Lipschitz constant $\theta < 1$. The MSA is often used for solving the nonlinear PDE of heat conduction [2]. The main idea of this approach is that firstly the nonlinear PDE of heat conduction can be transformed into a nonlinear second-order ODE for independent variable η, by means of the classical Boltzmann and similar transformations. The obtained ODE can be solved numerically by the method of successive approximations [2]. On the other hand, since the beginning of the 1980s, Adomian's decomposition method has been applied to a wide class of functional equations [3-5]. ADM gives the solution as an infinite series usually converging to an accurate solution. There is a closed relationship between these two methods. We consider this connection for heat equation and Burger's equation.

2 MSA for heat equation

We consider the initial value problem

$$\frac{\partial u}{\partial t} = A(x,y,z,t)\frac{\partial^2 u}{\partial x^2} + B(x,y,z,t)\frac{\partial^2 u}{\partial y^2} + C(x,y,z,t)\frac{\partial^2 u}{\partial z^2} + D(x,y,z,t), \tag{2.1}$$

$$u(x,y,z,0) = f(x,y,z). \tag{2.2}$$

Integrating Eq2.1 w.r.t t on the interval $(0,t)$, we obtain

$$u(x,y,z,t) = u(x,y,z,0) + \int_0^t D(x,y,z,\xi)d\xi + \int_0^t \left(A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial y^2} + C\frac{\partial^2}{\partial z^2}\right)ud\xi. \tag{4'}$$

The MSA (1.3) for (4') looks like

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,0)+\int_0^t D(x,y,z,\xi)d\xi+\int_0^t \left(A\frac{\partial^2}{\partial x^2}+B\frac{\partial^2}{\partial y^2}+C\frac{\partial^2}{\partial z^2}\right)u_n d\xi, \quad n=0,1,\ldots \tag{2.3}$$

with a given starting value $u_0(x,y,z,t)$. Computation by formula (2.3) can be performed by using symbolic computation packages Maple or Mathematica. On the other hand, Adomian decomposition method decompose the solution $u(x,y,z,t)$ by an infinite series of components

$$u=\sum_{k=0}^{\infty} v_k(x,y,z,t), \tag{2.4}$$

where the components v_k are determined recursively as [4]:

$$v_0 = f(x,y,z)+\int_0^t Dd\xi, \tag{2.5}$$

$$v_{k+1} = \int_0^t \left(A\frac{\partial^2}{\partial x^2}+B\frac{\partial^2}{\partial y^2}+C\frac{\partial^2}{\partial z^2}\right)v_k d\xi,\ k=0,1,\ldots. \tag{2.6}$$

If we choose $u_0(x,y,z,t)$ in (2.3) as

$$u_0(x,y,z,t) = f(x,y,z)+\int_0^t Dd\xi \equiv v_0,$$

then from (2.3) for $n=0$, we obtain

$$u_1(x,y,z,t) = u_0(x,y,z,0)+\int_0^t Dd\xi+\int_0^t \left(A\frac{\partial^2}{\partial x^2}+B\frac{\partial^2}{\partial y^2}+C\frac{\partial^2}{\partial z^2}\right)u_0(x,y,z,\xi)d\xi = v_0+v_1.$$

Analogously, from (2.3) for $n=1$, using (2.5), (2.6), we obtain

$$\begin{aligned} u_2(x,y,z,t) &= u1(x,y,z,0)+\int_0^t Dd\xi+\int_0^t \left(A\frac{\partial^2}{\partial x^2}+B\frac{\partial^2}{\partial y^2}+C\frac{\partial^2}{\partial z^2}\right)u_1(x,y,z,\xi)d\xi = \\ & v_0+\int_0^t \left(A\frac{\partial^2}{\partial x^2}+B\frac{\partial^2}{\partial y^2}+C\frac{\partial^2}{\partial z^2}\right)(v_0+v_1)d\xi = v_0+v_1+v_2. \end{aligned}$$

By induction one can prove that

$$u_n(x,y,z,t) = \sum_{k=0}^{n} v_k(x,y,z,t). \tag{2.7}$$

It means that MSA and Adomian's decomposition method give identical results. We apply the MSA to solve the following problem:

Example 1. Let us solve the following initial value problem [4]

$$\frac{\partial u}{\partial t} = \frac{1}{6}\left(x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}\right),$$
$$u(x,y,z,0) = x^2y^2z^2. \tag{2.8}$$

The equivalent equation for (2.8) is

$$u(x,y,z,t) = u(x,y,z,0) + \frac{1}{6}\int_0^t \left(x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}\right) d\xi.$$

The MSA for this equation looks like:

$$u_{n+1}(x,y,z,t) = u_n(x,y,z,0)+\frac{1}{6}\int_0^t \left(x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2} + z^2\frac{\partial^2}{\partial z^2}\right) u_n(x,y,z,\xi)d\xi,\ n = 0,1,\ldots$$

with $u_0(x,y,z,t) = x^2y^2z^2$. From this we obtain

$$u_1(x,y,z,t) = x^2y^2z^2 + x^2y^2z^2t = x^2y^2z^2(1+t),$$
$$u_2(x,y,z,t) = x^2y^2z^2(1+t+\frac{t^2}{2}),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$u_n(x,y,z,t) = x^2y^2z^2\left(1+t+\frac{t^2}{2}+\cdots+\frac{t^n}{n!}\right).$$

Consequently, we have

$$\lim_{n\to\infty} u_n(x,y,z,t) = x^2y^2z^2e^t.$$

Thus we obtain the exact solution of (2.8). The MSA was also successfully applied to Laplace equation [6].

3 MSA for Burger's equation

A well-known model is Burger's equation

$$u_t + uu_x = \nu u_{xx},\ a < x < b,\ t > 0, \tag{3.1}$$

which is one-dimensional quasi-linear parabolic partial differential equation and $\nu > 0$ is the coefficient of the kinematics viscosity of the fluid. We consider Eq.(3.1)with the following initial and boundary conditions:

$$u(x,0) = f(x),\ a < x < b,$$

$$u(a,t) = g_1(t),\ u(b,t) = g_2(t),\ t > 0.$$

Here we consider two problems.

Problem 1. Let us consider Burger's equation (3.1) with initial and homogeneous boundary conditions

$$u(x,0) = \sin \pi x,\ 0 < x < 1,$$

$$u(0,t) = u(1,t) = 0,\ t > 0. \tag{3.2}$$

It is well known that by Hopf-Cole transformation $u = -2\nu\theta_x/\theta$, subject to (3.2) transforms to

$$\theta_t = \nu\theta_{xx} \tag{3.3}$$

$$\theta_0(x) = \theta(x,0) = \exp\left\{-\frac{1-\cos\pi x}{2\pi\nu}\right\},\ 0 < x < 1$$

$$\theta_x(0,t) = \theta_x(1,t) = 0,\ t > 0. \tag{3.4}$$

The Fourier series solution of (3.3) and (3.4) is [3]

$$\theta(x,t) = a_0 + \sum_{m=1}^{\infty} a_m e^{-m^2\pi^2\nu t}\cos(m\pi x), \tag{3.5}$$

where a_i are Fourier coefficients and

$$a_0 = \int_0^1 \exp\{-(2\pi\nu)^{-1}[1-\cos(\pi x)]\}dx, \tag{3.6}$$

$$a_m = 2\int_0^1 \exp\{-(2\pi\nu)^{-1}[1-\cos(\pi x)]\}\cos(m\pi x)dx, \ m \geq 1.$$

The exact solution to the problem 1 is [2]

$$u(x,t) = \frac{2\pi\nu\sum_{m=1}^{\infty} a_m e^{-m^2\pi^2\nu t}\cdot m\cdot\sin(m\pi x)}{a_0+\sum_{m=1}^{\infty} a_m e^{-m^2\pi^2\nu t}\cos(m\pi x)}. \tag{3.7}$$

Problem 2. The second problem is Burger's equation (3.1) with the initial condition

$$u(x,0) = 4x(1-x), \ 0 < x < 1$$

and homogeneous boundary condition $u(0,t) = u(1,t) = 0$. The Fourier series solution coefficient is

$$a_0 = \int_0^1 \exp\{-x^2(3\nu)^{-1}(3-2x)\}dx,$$

$$a_m = \int_0^1 \exp\{-x^2(3\nu)^{-1}(3-2x)\}\cos(m\pi x)dx, \ m \geq 1.$$

The equation (3.3) is a particular case of (2.1) with $A = \nu$, $B = C = D = 0$ and therefore the MSA and Adomian decomposition method will give identical results. For numerical solution of Burger's equation have been often used the finite-difference methods, Adomian's decomposition method [3,4] and restrictive Taylor approximation [5] and so forth.

Now we consider the MSA for (3.3) and (3.4). The equivalent equation for (3.3) is

$$\theta(x,t) = \theta(x,0) + \nu\int_0^t \theta_{xx}(x,\xi)d\xi$$

and the MSA for last equation is

$$\theta_{n+1}(x,t) = \theta_n(x,0) + \nu\int_0^t \theta_{nxx}(x,\xi)d\xi, \ n = 0,1,\ldots \tag{3.8}$$

with initial approximation $\theta_0(x,t) = \theta_0(x)$. From (3.8), for $n = 0$, we get

$$\theta_1(x,t) = \theta_0(x) + \nu\theta_0''(x)\cdot t; \ \frac{\partial^2}{\partial x^2}\theta_1(x,t) = \theta_0''(x) + \nu\theta_0^{(IV)}(x)\cdot t. \tag{3.9}$$

Analogously, from (3.8) and (3.9), we get

$$\theta_2(x,t) = \theta_0(x) + \nu\theta_0^{(2)}t + \frac{\theta_0^{(IV)}}{2}(\nu t)^2.$$

From this we find

$$\theta_2(x,0) = \theta_0(x),$$

$$\frac{\partial^2}{\partial x^2}\theta_2 = \theta_0^{(2)} + \nu\theta_0^{(4)} + \frac{\theta_0^{(6)}}{2}(\nu t)^2.$$

Using last expressions, from (3.8), for $n = 2$, we get

$$\theta_3(x,t) = \theta_0(x) + \nu\theta_0^{(2)}(x)\cdot t + \frac{\theta_0^{(4)}}{2}(\nu t)^2 + \frac{\theta_0^{(6)}}{3!}(\nu t)^3.$$

Repeating the above process, we obtain

$$\theta_n(x,t) = \sum_{k=0}^{n}\frac{\theta_0^{(2k)(x)}}{k!}(\nu t)^k, \quad n = 1,2,\ldots \tag{3.10}$$

From (3.10) it is easy to show that

$$\theta_n(x,0) = \theta_0(x); \quad \theta_{nx}(0,t) = \theta_{nx}(1,t) = 0,$$

i.e., the n-th approximation (3.10) for all $n \geq 1$ satisfies the initial and boundary conditions (3.4). Obviously, if the sequence $\{\theta_n(x,t)\}$ has a limit, then it will be a solution of Eq.(3.3). The n-th approximation for Burger's equation (3.1) is

$$u_n(x,t) = -2\nu\frac{\sum_{k=0}^{n}\theta_0^{2k+1}(x)\frac{(\nu t)^k}{k!}}{\sum_{k=0}^{n}\theta_0^{2k}(x)\frac{(\nu t)^k}{k!}} \tag{3.11}$$

On the other hand, from (3.5) we have

$$\theta_0(x) = \theta(x,0) = a_0 + \sum_{m=1}^{\infty}a_m cos(m\pi x). \tag{3.12}$$

Differentiating Eq.(3.12) $2k$-times, we get

$$\theta_0^{(2k)}(x) = \sum_{m=1}^{\infty}a_m(-1)^k(m\pi)^{2k}cos(m\pi x). \tag{3.13}$$

Substituting (3.13) into (3.10), we have

$$\theta_n(x,t) = \theta_0(x) + \sum_{k=1}^{n} \frac{\theta_0^{(2k)}(x)}{k!}(\nu t)^k = a_0 + \sum_{m=1}^{\infty} a_m \cos(m\pi x) + \sum_{m=1}^{\infty} a_m \cos(m\pi x)\left(\sum_{k=0}^{n} \frac{(-1)^k (m^2\pi^2\nu t)^k}{k!} - 1\right) = a_0 + \sum_{m=1}^{\infty} a_m \cos(m\pi x)\left(\sum_{k=0}^{n} \frac{(-m^2\pi^2\nu t)^k}{k!}\right). \tag{3.14}$$

It is clear that

$$\lim_{n\to\infty} \theta_n(x,t) = a_0 + \sum_{m=1}^{\infty} a_m \exp\{-m^2\pi^2\nu t\} \cos(m\pi x) = \theta(x,t),$$

i.e., we obtain the exact solution of Eq.(3.3). Thereby from (3.11), we get

$$\lim_{n\to\infty} u_n(x,t) = u(x,t).$$

Thus, the MSA gives the exact solution of problem 1. It should be mentioned that in [3] the exact solution of problem 1 can not find by Adomian's decomposition method and has to use the modified Adomian's method. As a result they obtained only approximate solution. Analogously the problem 2 was solved by MSA as the problem 1 and, it gives in the limit the exact solution (3.7) with coefficient (3). From (3.14) it is clear that the $n-$th approximation $u_n(x,t) = -2\nu\frac{\theta_{nx}(x,t)}{\theta_n(x,t)}$ will give the solution of Burger's equation with high accuracy and has rapid convergence.

4 Conclusions

The main goal of this paper has been to obtain an analytical solution for the heat and Burger's equation. We have achieved this goal by applying MSA. We have also shown that MSA and Adomian's decompositions method are equivalent for heat and Burger's equation. The $n-$th approximation $u_n(x,t)$ will give the solution of Burger's equation with high accuracy even for relatively a few small n and has a rapid convergence. Thus MSA introduces a significant improvement in solving Burger's equation over existing methods.

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