On Hereditarily Baire Space

Batkhuu Tserennadmid
School of Mathematic and Statistics of Mongolian State University of Education
Email: tsbatkhuu@yahoo.com

Keywords: α_0-box topology product of topological spaces, strong Choquet space, hereditarily Baire space

Abstract

In this note we give a definition of α_0-box topology product of topological spaces and using this concept we prove that there exists a strong Choquet space and regular topological space which not contains closed subspace homeomorphic to rational numbers $\mathbb{Q}$ but both are not hereditarily Baire spaces.

1 Introduction

We start with some definitions and well-known results:

Let X be topological space. A set $A \subseteq X$ is called nowhere dense if its closure $\overline{A}$ has empty interior, i.e., $Int(\overline{A}) = \emptyset$.

A set $A \subseteq X$ is meager (of first category) if $A = \bigcup_{n \in \mathbb{N}} A_n$, where A_n is nowhere dense. A non-meager set is also called of a second category. The complement of a meager set is called comeager.

A topological space X is called a Baire space if it satisfies following equivalent conditions:

(i). Every non-empty open set in X is non-meager.

(ii). Every comeager set in X is dense.

(iii). The intersection of countably many dense open sets in X is dense.

A Baire space is called hereditarily Baire if every closed subspace of X is a Baire.

Let X be non-empty topological space. The Choquet game G_X of X is defined as follows: Players I and II take turns in playing non-empty open subsets of X

$$\begin{array}{cccccc} I & U_0 & & U_1 & & \dots \\ II & & V_0 & & V_1 & \dots \end{array}$$

so that $U_0 \supset V_0 \supset U_1 \supset V_1 \supset \dots$.

We say that II wins this run of the game if $\bigcap_n V_n (= \bigcap_n U_n) \neq \emptyset$. Thus I wins if $\bigcap_n U_n (= \bigcap_n V_n) = \emptyset$.

A strategy for I in this game is a "rule" that tells him how to play, for each n, his n-th move U_n, given II's previous moves $V_0, V_1, \dots, V_{n-1}$.

Oxtoby theorem: A non-empty topological space X is Baire space iff player I has no winning strategy in the Choquet game G_X.

A non-empty topological space X is called a Choquet space if player II has a winning strategy in the Choquet game G_X.

Since it is not possible for both players to have a winning strategy in G_X,it follows that every Choquet space is Baire. The converse fails even for non-empty separable metrizable spaces, using the axioms of Choice.

Given a non-empty topological space X, the strong Choquet game G_X^s is defined as follows: Players I and II take turns in playing non-empty open subsets of X a

$$\begin{array}{llllll} I & x_0, U_0 & & x_1, U_1 & & \dots \\ II & & V_0 & & V_1 & \dots \end{array}$$

Players I and II take turns in playing non-empty open sets of X as a in Choquet game, but additionally I is required to play a poin $x_n \in U_n$ and II must then play $V_n \subset U_n$ with $x_n \in V_n$. So we must have $\quad U_0 \supset V_0 \supset U_1 \supset V_1 \supset \dots, \quad x_n \in U_n, V_n$.

Player II wins this run of the game $\bigcap_n V_n \, (= \bigcap_n U_n) \neq \emptyset$. Thus I wins if $\bigcap_n U_n \, (= \bigcap_n V_n) = \emptyset$. A non-empty topological space X is called a strong Choquet space if player II has a winning strategy in the strong Choquet game G_X^s.

It is obvious that any strong Choquet space is Choquet. About of a connection between a hereditarily Baire space and a strong Choquet spaces in class of metric spaces are known following theorems:

A metric space X is a strong Choquet space if and only if it is a complete metric space.(Choquet)

A metric space X is a hereditarily Baire space if and only if player I have no winning strategy in G_X^s. Particularly, from it follows that every metric Choquet space is hereditarily Baire.(Debs)

A first countable regular space is a hereditarily Baire if and only if which not contains a closed subspace homeomorphic to rational numbers $\mathbb{Q}$.(Hurewicz-van Jeroen Doumen)

2 α_0-box topology and a construction of the strong Choquet and the regular topological spaces which not contains closed subspace homeomorphic to rational numbers $\mathbb{Q}$ but both are not hereditarily Baire spaces

Definition 1. Suppose we are given a family $\{X_i\}_{i\in I}$ of topological spaces and consider the Cartesian product $\prod\limits_{i\in I} X_i$ and the family of mappings p_A, $A-$countable set of I, where p_A assigns to the point $x = (x_i)_{i\in I} \in X = \prod_{i\in I} X_i$ it's A-th side $x = (x_i)_{i\in A}$.

The set $X = \prod_{i\in I} X_i$ with the topology generalized by the family of mappings $p_A, A-$ countable set of I is called α_0-box topology product of the spaces $X_i, i \in I$ and denoted by $\bigotimes_{i\in I} X_i$.

If we define the game $G^s_{X,\beta}$ as a Choquet game but only difference is choosing open subset of X from it's open base β than it is not hard to see following lemma.[2]

Lemma 1. Let β_1, β_2 are bases of topological space X. Than following conditions are equivalent.

(i). Player I(Player II) has a winning strategy in G^s_{X,β_1}.

(ii). Player I(Player II) has a winning strategy in G^s_{X,β_2}.

Theorem 1. Any α_0-box topology product of strong Choquet spaces is a strong Choquet space.

Proof. By the previous lemma it is suffices to prove that player II has a winning strategy in $G^s_{X,\beta}$ game.

Let σ_i is a player II's winning strategy in $G^s_{X_i}$. Suppose player I's n-th move is (x_n, U_n), where $U_n = \bigotimes_{i\in I} U_{n_i}$.

Let's define player II's strategy

$$\sigma\left((x_1, U_1), V_1, \ldots, (x_n, U_n)\right) = V_n = \bigotimes_{i\in I} V_{n_i}$$

, where

$$V_{n_i} = \begin{cases} V_{n_i} = \sigma_i\left((x_{1_i}, U_{1_i}), V_1, \ldots, (x_{n_i}, U_{n_i})\right), & if \quad U_{n_i} \neq X_i, \\ X_i, \quad if \quad U_{n_i} = X_i \end{cases}$$

Now if we remember that each σ_i is a player II's winning strategy in G^s_X then $\bigcap_n U_n \neq \emptyset$ or σ is player II's winning strategy in $G^s_{X,\beta}$. □

Corollary 1. There exists a strong Choquet space which is not hereditarily Baire space.

Proof. Suppose that X is α_0-box topology product space of topological spaces $\{X_i\}_{i\in I}$, where X_i coincides with real line $\mathbb{R}$.

From Theorem 1 it follows that X is strong Choquet space. Now we shall show that X is not a hereditarily Baire space. It suffices to point a closed subset Z in X, which is meager (set of first category)in itself.

Consider $Z = \{x = (x_i)_{i\in I} \in X \mid \quad |\{i \in I| \quad x_i \neq 0\}| < \alpha_0$.

At first let us show that Z is closed subset in X. Let $x = (x_i)_{i\in I} \in X \setminus Z$. Then there exists a countable set $A \subset I$ such that $x_i \neq 0$ for every $i \in A$. Let us take a non-empty neighbourhood U_i of x_i. Then open set $V = \bigotimes_{i\in I} V_i$ in which $V_i = U_i$ if $i \in A$ and $V_i = \mathbb{R}$ if $i \in I \setminus A$ will be a neighbourhood of $x = (x_i)_{i\in I}$ and $V \bigcap Z = \emptyset$.

Now let us prove that Z is meager set.

For every $n < \alpha_0$ consider $Z_n = \{x = (x_i)_{i\in I} \in X| \quad |\{i \in I \mid \quad x_i \neq 0\}| \leq n\}$. It is obvious that $Z_n \subset Z_{n+1}$ and $Z = \bigcup_n Z_n$ and Z_n closed in Z.

Let $V = \bigotimes_{i \in I} V_i$ is a base element such that $U = V \bigcap Z \neq \emptyset$. Let $n_0 = |\{i \in I \mid 0 \notin V_i\}|$. Then for every $n \leq n_0$ there exists a point $x \in U$ such that $x \notin Z_n$. Since Z_n nowhere dense in Z for every $n \leq n_0$ or Z is set of first category. $\square$

Definition 2. Let $X = \bigotimes_{i \in I} X_i$ and $p_A : \bigotimes_{i \in I} X_i \mapsto \bigotimes_{i \in A} X_i$ projection map. If $Y \subset X$ and $p_{I \setminus A}(x) = p_{I \setminus A}(y)$ for any $x, y \in Y$ then we call Y depends on A coordinates.

It is not hard to prove next lemma.[2]

Lemma 2. Let $X = \bigotimes_{i \in I} X_i$ and $Y \subset X$, $A \subset I$. The following conditions are equivalent.

(i). Y depends on A coordinates and $A \subset B$ then Y depends on B coordinates.

(ii). $Y = \bigcup_{n=1}^{k} Y_n$ and every Y_n depends on A coordinates then Y depends on A coordinates.

(iii). Y depends on A coordinates then $p^{-1}_{I \setminus A}\left(p_{I \setminus A}(Y)\right)$ contains Y and depends on A coordinates. Also $p^{-1}_{I \setminus A}\left(p_{I \setminus A}(Y)\right)$ closed subset in X and homeomorphic to $\bigotimes_{i \in A} X_i$.

Lemma 3. Let $X = \bigotimes_{i \in I} X_i$, $x \in X$ and a sequence of sets $K_n \subset X$ converges to x,i.e., there exists a number n_0 such that K_n contained in every neighbourhood V of x for every $n > n_0$. Then there exists a finite set $H \subset I$ and number n_0 such that K_n depends on H coordinates for every $n > n_0$.

Proof. Suppose that there not exists such a set and number. Then we can find the sequence of pairwise distinctive indexes $i_n \in I$ and $x^n = (x_i^n)_{i \in I} \in K_n$ points such that $d(X_i) < 2^{\alpha_0}$. From it follows that there exists a neighbourhood V of x

which not contains x_n for every n. Contradiction. $\square$

Corollary 2. There exists a regular topological space X which not contains closed subspace homeomorphic to rational numbers $\mathbb{Q}$ but that is not a hereditarily Baire space.

Proof. Suppose that $X = \bigotimes_{i \in I} X_i$ where every X_i coincides with real line $\mathbb{R}$.

Consider there exists closed subspace homeomorphic to raional numbers $\mathbb{Q}$ in X. For every point $x \in Z$ let us take a fundamental neighbourhood system $U_n(x)$, consisting of open-closed sets in Z such that $U_{n+1}(x) \subset U_n(x)$.

It is obvious that $U_n(x)$ converges to x. By the Lemma 3. there exists a finite set $A \subset I$ and a number n_0 such that $U_{n_0}(x)$ depends on A coordin ates and $U_{n0(x)}$ homeomorphic to closed subspace in $X = \bigotimes_{i \in I} \mathbb{R}_i$ according to Lemma 2.3. Clearly, $U_{n_0}(x)$ does not contain isolated points, then homeomorphic to rational numbers $\mathbb{Q}$. That is contradiction to the completeness of A. $\square$

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